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The Fourier-spectrum of circular sine and cosine gratings with arbitrary radial phases

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Abstract

The Fourier spectra of circular gratings having sine or cosine radial profiles are derived, and their particular properties are discussed. These results are then extended to the most general form of circular sinusoidal gratings, namely: circular sine or cosine gratings with any arbitrary radial phase. © 1998 Elsevier Science B.V.

1. Introduction

Circular sine and cosine gratings are defined as radially periodic functions whose radial profiles are, respectively, $\sin(2\pi fr)$ or $\cos(2\pi fr)$, $r \geq 0$ (see Fig. 1a and Fig. 2a). These functions can be seen also as the 2D surface which is obtained by revolving the positive x-direction of the 1D function $\sin(2\pi fx)$ (respectively: $\cos(2\pi fx)$) about the vertical axis. These functions represent in optics circular waves which emanate from a point source, or simple circular gratings; their Fourier transforms may arise, for example, in connection with the Fraunhofer diffraction pattern generated by these circular gratings. However, in spite of the simple appearance of these functions, their Fourier transforms cannot be found in standard tables of Fourier (or Hankel) transform pairs; the reasons for this fact will become clear below. The derivation of these Fourier transforms is, therefore, the main aim of the present paper.

Our work on this subject was motivated by our research on the Fourier spectrum of radially periodic images. In a previous paper describing our first results in this direction [1], we already derived the Fourier spectrum of the circular cosine function; this result is briefly presented here in Section 2. Then, in order to complete the picture, we derive in Section 3 the Fourier spectrum of the circular sine function, and we compare its particular properties with those of its cosine counterpart. Then, based on these results, we derive in Section 4 the Fourier spectrum of the most general circular sinusoidal functions, namely: $g(r) = \sin[2\pi f(r+\varphi)]$ or $g(r) = \cos[2\pi f(r+\varphi)]$, $r \geq 0$, with an arbitrary constant radial phase φ, and we show how these spectra gradually evolve when the constant φ is being varied throughout one full period of the sinusoidal function, say, between 0 and $T = 1/f$.

Note that the Fourier transform conventions throughout this paper are based on Bracewell's notations; thus, the Fourier transform of a function $f(x,y)$ is given by [2]:

$$F(u,v) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(x,y)\exp(-i2\pi(ux+vy))\,dx\,dy$$

and

$$f(x,y) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} F(u,v)\exp(i2\pi(ux+vy))\,du\,dv.$$

Similarly, the Hankel transform, which is an equivalent way to express the 2D Fourier transform of a circularly symmetric function $g(r) = f(x,y)$ based on its radius $r = \sqrt{x^2+y^2}$, is given according to Bracewell's notations by [3]:

$$G(q) = 2\pi\int_0^{\infty} g(r)\,J_0(2\pi qr)\,r\,dr$$

and

$$g(r) = 2\pi\int_0^{\infty} G(q)\,J_0(2\pi qr)\,q\,dq,$$

where $r = \sqrt{x^2+y^2} \geq 0$ and $q = \sqrt{u^2+v^2} \geq 0$. For the

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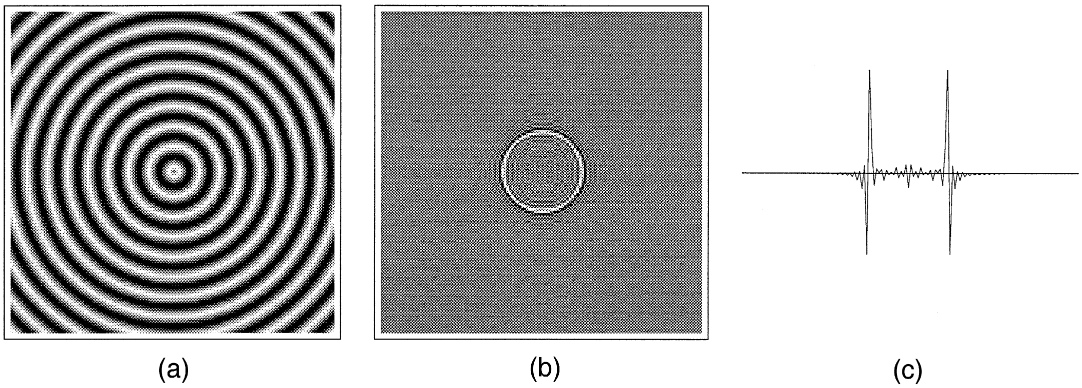

Fig. 1. (a) The circular sine grating $\sin(2\pi fr)$. (b) Its Fourier spectrum as obtained by a two-dimensional DFT. (c) The average cross section through the origin of this DFT (averaged through all directions $\theta = 0°, \ldots, 360°$, in order to compensate for local DFT artifacts).

sake of convenience, the Hankel transform is also denoted more compactly by $\mathscr{H}[g(r)] = G(q)$ and $\mathscr{H}^{-1}[G(q)] = g(r)$, or, in a more symmetric way, by: $g(r) \overset{\mathscr{H}}{\leftrightarrow} G(q)$.

2. The spectrum of the circular cosine grating

The circular cosine grating can be expressed either in Cartesian coordinates:

$$g(x,y) = \cos\left(2\pi f\sqrt{x^2+y^2}\right) \tag{1}$$

or in polar coordinates:

$$g(r) = \cos(2\pi fr), \tag{2}$$

where f denotes the radial frequency. As we have shown in Ref. [1], the closest 'hint' one can find in the literature for the Fourier spectrum of this function is hidden in the following general Fourier (or rather Hankel) transform pair [4]:

$$\frac{1}{r^{\mu}} J_{\mu}(2\pi fr) \overset{\mathscr{H}}{\leftrightarrow} \frac{\pi^{\mu-1}}{f^{\mu}\Gamma(\mu)}(f^2-q^2)^{\mu-1}\,\mathrm{rect}\left(\frac{q}{2f}\right), \tag{3}$$

where $\mathrm{rect}(q/2f)$ means truncation to zero beyond the circle of radius f, and

$$r = \sqrt{x^2+y^2} \geq 0, \quad q = \sqrt{u^2+v^2} \geq 0. \tag{4}$$

This general formula gives several interesting Hankel transform pairs for various values of μ. In particular, for $\mu = -1/2$ it gives the Hankel transform of the circular cosine (2), since [5]

$$\cos r = \sqrt{\pi r/2}\, J_{-1/2}(r).$$

It should be noted that transform pair (3) is usually given in the literature only for $\mu > 0$ (probably since for

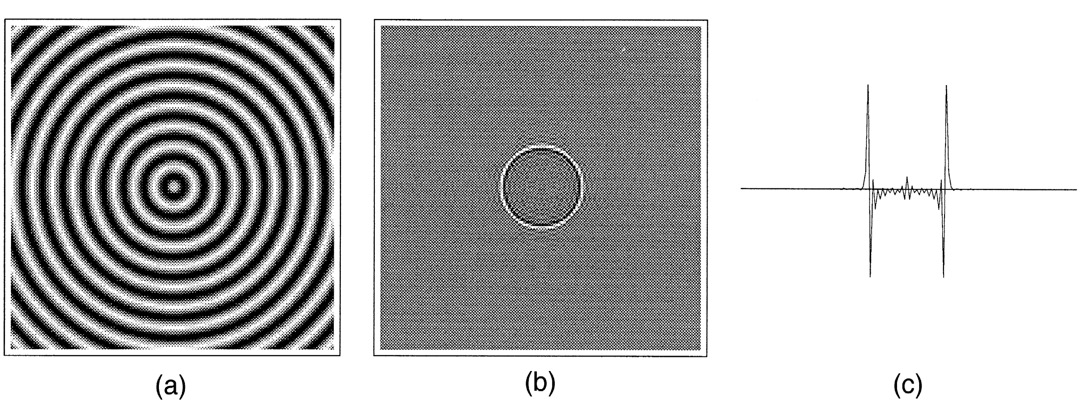

Fig. 2. (a) The circular cosine grating $\cos(2\pi fr)$. (b) Its Fourier spectrum as obtained by a two-dimensional DFT. (c) The average cross section through the origin of this DFT (averaged through all directions $\theta = 0°, \ldots, 360°$, in order to compensate for local DFT artifacts).

$\mu \le 0$ the functions on the left hand side of (3) do not properly decay, and consequently their Hankel transforms include a 'wild' (impulsive) behaviour on the border of their circular spectrum support). However, according to Ref. [6] formula (3) is also valid for non-integer negative values of μ, including our case of $\mu = -1/2$. For $\mu = -1/2$ formula (3) gives, therefore (remembering that $\Gamma(-1/2) = -2\sqrt{\pi}$):

$$\cos(2\pi fr) \overset{\mathscr{H}}{\leftrightarrow} -\frac{f}{2\pi}\frac{1}{(f^2-q^2)^{3/2}}\,\mathrm{rect}\left(\frac{q}{2f}\right) \quad (r,q \ge 0). \tag{5}$$

The Fourier spectrum of the circular cosine (1), as obtained by 2D DFT (see Fig. 2b, 2c), confirms this result for the interior of the ring, $\{q < f\}$. However, it also indicates, as we have indeed expected, that the behaviour of the spectrum on its *singular support*, $\{q = f\}$, is more complicated: in addition to the negative peak at the internal side of the ring, as predicted by Eq. (5), it clearly shows also a positive impulsive behaviour at the external side of the ring, so that a vertical section through the spectrum origin would look like in Fig. 4. Note that the external impulsive border of this ring is sharp, whereas the internal, negative peak of the ring is characterized by a smooth decay transition in the form of a continuous wake. As we have shown in Ref. [1] this peculiar impulsive behaviour is not an artifact due to the limitations of the discrete Fourier transform in representing the non-decaying circular cosine function (1), and it indeed represents an inherent feature of this Fourier transform. Moreover, we have shown there that this peculiar impulsive behaviour corresponds to that of the half-order derivative of the Dirac impulse $\delta(x)$, which is given in the literature by:

$$\delta^{(1/2)}(x) = -\frac{1}{2\sqrt{\pi}}\frac{1}{x^{3/2}}\,\mathrm{step}(x) \tag{6}$$

as a special case of the general formula [7]:

$$\delta^{(\lambda)}(x) = \frac{1}{\Gamma(-\lambda)}\frac{1}{x^{\lambda+1}}\,\mathrm{step}(x), \tag{7}$$

where step(x) is defined as 0 for $x < 0$ and 1 for $x > 0$.

As shown in Appendix B of Ref. [1], Eq. (6) describes the properties of $\delta^{(1/2)}(x)$ to the right of $x = 0$; but when we approach $\delta^{(1/2)}(x)$ by a sequence of functions which are defined to both sides of $x = 0$, it becomes apparent that $\delta^{(1/2)}(x)$ has at the point $x = 0$ a peculiar impulsive behaviour: it has a positive impulsive peak at the left-hand side of $x = 0$, while to the right-hand side of $x = 0$ it has a negative peak, which smoothly decays in the form of a negative continuous wake trailing off asymptotically to the positive direction of the x-axis (see Fig. 3). The connection between these properties of $\delta^{(1/2)}(x)$ and the peculiar impulsive behaviour of the spectrum of $\cos(2\pi fr)$ can be seen now from the following expression, which is obtained from Eq. (6):

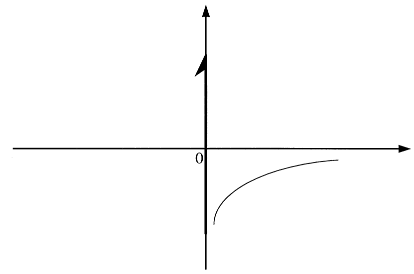

Fig. 3. Schematic plot of $\delta^{(1/2)}(x)$, the half-order derivative of the impulse $\delta(x)$.

$$\delta^{(1/2)}(f^2-q^2) = -\frac{1}{2\sqrt{\pi}}\frac{1}{(f^2-q^2)^{3/2}}\,\mathrm{rect}\left(\frac{q}{2f}\right) \tag{8}$$

(note the truncation *beyond* the radius f, due to the inside-out inversion of $\delta^{(1/2)}(f^2-q^2)$ with respect to $\delta^{(1/2)}(q^2-f^2)$). Using Eq. (8) the spectrum (or the Hankel transform) of the circular cosine function can be expressed in terms of $\delta^{(1/2)}(\ )$ as follows, thus emphasizing its impulsive behaviour:

$$\cos(2\pi fr) \overset{\mathscr{H}}{\leftrightarrow} \frac{f}{\sqrt{\pi}}\,\delta^{(1/2)}(f^2-q^2). \tag{9}$$

This expression can be further simplified by expressing $\delta^{(1/2)}(f^2-q^2)$ in terms of the half-order derivative of the simple impulse ring, $\delta^{(1/2)}(f-q)$: It is known that for any integer k and constant $c > 0$ there exists the relation [8]:

$$\delta^{(k)}(r^2-c^2) = \frac{1}{(r+c)^{k+1}}\,\delta^{(k)}(r-c).$$

However, this relation turns out to be valid also for non-integer values of k, denoted below by λ, since by Eq. (7) we have (for $r,c > 0$):

$$\begin{aligned}\delta^{(\lambda)}(r^2-c^2) &= \frac{1}{\Gamma(-\lambda)}\frac{1}{(r^2-c^2)^{\lambda+1}}\,\mathrm{step}(r^2-c^2)\\ &= \frac{1}{\Gamma(-\lambda)}\frac{1}{(r+c)^{\lambda+1}(r-c)^{\lambda+1}}\\ &\quad\times\mathrm{step}(r-c)\\ &= \frac{1}{(r+c)^{\lambda+1}}\,\delta^{(\lambda)}(r-c).\end{aligned} \tag{10}$$

And hence, since $\delta^{(1/2)}(-x)$ is the mirror-image of $\delta^{(1/2)}(x)$:

$$\delta^{(1/2)}(c^2-r^2) = \frac{1}{(r+c)^{3/2}}\,\delta^{(1/2)}(c-r).$$

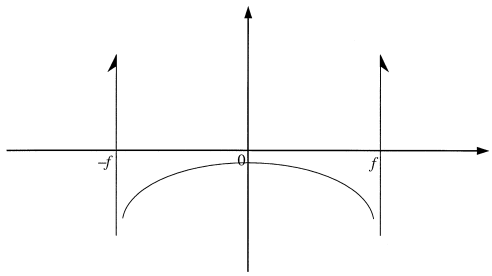

Fig. 4. Schematic plot of a cross section through the spectrum of the circular cosine function $\cos(2\pi fr)$. Notice the positive impulse on the external border of the ring.

Therefore we obtain the following expression for the spectrum of the circular cosine with radial frequency $f > 0$:

$$\cos(2\pi fr) \overset{\mathscr{H}}{\leftrightarrow} \frac{f}{\sqrt{\pi}} \frac{1}{(f+q)^{3/2}} \delta^{(1/2)}(f-q). \tag{11}$$

Note that in the degenerate case of $f = 0$, in which the cosine becomes identically 1, the spectrum is given by the well-known transform pair [9]:

$$1 \overset{\mathscr{H}}{\leftrightarrow} \frac{1}{\pi |q|} \delta(q).$$

Eq. (11) indeed confirms observations by 2D DFT which clearly show that as f increases – not only the radius of the impulse ring increases, but also its wake becomes weaker. Note that $\delta^{(1/2)}(f-q)$ is the inside-out inverted counterpart of the $\delta^{(1/2)}(q-f)$ ring, where the negative wake trails off inwards, towards the centre, and the positive impulsive peak is located in the outer side (see Fig. 4).

3. The spectrum of the circular sine grating

The circular sine grating is expressed in Cartesian coordinates by:

$$g(x,y) = \sin\left(2\pi f \sqrt{x^2+y^2}\right) \tag{12}$$

and in polar coordinates

$$g(r) = \sin(2\pi fr), \tag{13}$$

where f denotes the radial frequency. Like in the case of the circular cosine function the Fourier spectrum of this function cannot be found in standard tables of Fourier (or Hankel) transform pairs. The closest 'hint' one can find in the literature for this case is hidden in another general Fourier (or rather Hankel) transform pair, which we adapt here from Ref. [10]:

$$r^{\mu} J_{\mu}(2\pi fr) \overset{\mathscr{H}}{\leftrightarrow} \frac{f^{\mu}}{\pi^{\mu+1}\Gamma(-\mu)} \frac{1}{(q^2-f^2)^{\mu+1}} \operatorname{step}\left(\frac{q}{2f}\right), \tag{14}$$

where $\operatorname{step}(q/2f)$ means truncation to zero inside the circle of radius f, and

$$r = \sqrt{x^2+y^2} \geq 0, \quad q = \sqrt{u^2+v^2} \geq 0. \tag{15}$$

Like formula (3), this general formula too gives several interesting Hankel transform pairs for various values of μ. In particular, for $\mu = 1/2$ it gives the Hankel transform of the circular sine (13), since [5]:

$$\sin r = \sqrt{\pi r/2}\, J_{1/2}(r).$$

It should be noted that transform pair (14) is usually given in the literature only for $-1 < \mu < 0$ (note that for $\mu \geq 0$ the functions on the left hand side of (14) do not properly decay, and consequently their Hankel transforms include a 'wild' (impulsive) behaviour on the border of their circular spectrum support). However, according to Ref. [6] formula (14) is also valid for non-integer values of μ, including our case of $\mu = 1/2$. For $\mu = 1/2$ formula (14) gives, therefore:

$$\sin(2\pi fr) \overset{\mathscr{H}}{\leftrightarrow} -\frac{f}{2\pi} \frac{1}{(q^2-f^2)^{3/2}} \operatorname{step}\left(\frac{q}{2f}\right) \times (r,q \geq 0). \tag{16}$$

Like in the case of $\cos(2\pi fr)$, the Fourier spectrum of the circular sine (12), as obtained by 2D DFT (see Fig. 1b, 1c), confirms this result for the exterior of the ring, $\{q > f\}$. However, it also indicates, as we have indeed expected, that the behaviour of the spectrum on its *singular support*, $\{q = f\}$, is more complicated: in addition to the negative peak at the external side of the ring, as predicted by Eq. (16), it clearly shows also a positive impulsive behaviour at the internal side of the ring, so that a vertical section through the spectrum origin would look like in Fig. 5. Note that the internal impulsive border of

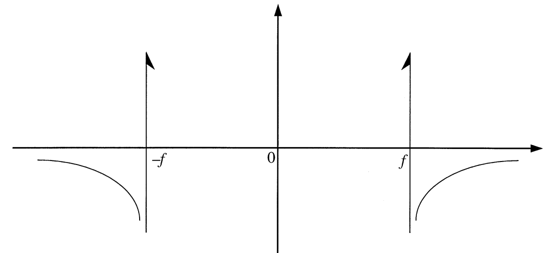

Fig. 5. Schematic plot of a cross section through the spectrum of the circular sine function $\sin(2\pi fr)$. Notice the positive impulse on the internal border of the ring.

this ring is sharp, whereas the external, negative peak of the ring is characterized by a smooth decay transition in the form of a continuous, asymptotic wake.

The connection between this peculiar impulsive behaviour of the spectrum of $\sin(2\pi fr)$ and the properties of $\delta^{(1/2)}(x)$ can be seen now from the following equation, which is obtained from Eq. (6):

$$\delta^{(1/2)}(q^2-f^2) = -\frac{1}{2\sqrt{\pi}}\frac{1}{(q^2-f^2)^{3/2}}\,\mathrm{step}\left(\frac{q}{2f}\right).$$

Using this equation, the spectrum of the sine function can be expressed in terms of $\delta^{(1/2)}(\;)$, emphasizing thus its impulsive behaviour:

$$\sin(2\pi fr) \overset{\mathscr{H}}{\leftrightarrow} \frac{f}{\sqrt{\pi}}\delta^{(1/2)}(q^2-f^2). \tag{17}$$

This expression can be further simplified by expressing $\delta^{(1/2)}(q^2-f^2)$ in terms of the half-order derivative of the simple impulse ring, $\delta^{(1/2)}(q-f)$, using Eq. (10).

We obtain, therefore, the following expression for the spectrum of the circular sine (with radial frequency $f>0$):

$$\sin(2\pi fr) \overset{\mathscr{H}}{\leftrightarrow} \frac{f}{\sqrt{\pi}}\frac{1}{(f+q)^{3/2}}\delta^{(1/2)}(q-f). \tag{18}$$

In the degenerate case of $f=0$ the sine becomes identically 0 and its spectrum vanishes, too.

By comparing Eqs. (11) and (18) we can see the remarkable symmetry between the Fourier spectra of the circular sine and cosine functions. In fact, the only difference between them is that in the spectrum of the circular sine function the impulse ring $\delta^{(1/2)}(q-f)$ is not inside-out inverted as in the case of the circular cosine function. This means that its negative wake trails off outwards, while the positive impulsive behaviour is located in the inner side of the ring (compare Figs. 4 and 5). (Note that unlike the simple impulse $\delta(x)$ which is symmetric, i.e. $\delta(-x)=\delta(x)$, and its derivative $\delta'(x)$ which is antisymmetric, i.e. $\delta'(-x)=-\delta'(x)$, $\delta^{(1/2)}(x)$ is *asymmetric*: $\delta^{(1/2)}(-x)$ is the mirror-image of $\delta^{(1/2)}(x)$.)

4. The spectra of the phase-shifted circular sine or cosine gratings

The most general form of a circular sinusoidal grating can be expressed in the form of a phase-shifted circular sine or cosine function, namely:

$$g(r)=\sin[2\pi f(r+\varphi)] \tag{19}$$

or

$$g(r)=\cos[(2\pi f(r+\varphi)], \tag{20}$$

where f denotes the radial frequency, φ is an arbitrary constant radial phase-shift, and $r\geq 0$. In fact, since these functions are radially periodic with radial period $T=1/f$, it is enough to consider the range of values $0\leq\varphi\leq 1/f$.

Let us find now the Fourier spectrum of the phase-shifted sine function (19). For this end we express the phase-shifted sine function in terms of sine and cosine functions with phase-shifts of $\varphi=0$, whose Fourier spectra we have already derived in Sections 2 and 3. This can be done by means of the well-known trigonometric identity $\sin(\alpha+\beta)=\sin\alpha\cos\beta+\cos\alpha\sin\beta$:

$$\begin{aligned}\sin[2\pi f(r+\varphi)] &= \sin(2\pi fr+2\pi f\varphi)\\ &= \cos(2\pi f\varphi)\sin(2\pi fr)\\ &\quad+\sin(2\pi f\varphi)\cos(2\pi fr).\end{aligned}$$

Since $\cos(2\pi f\varphi)$ and $\sin(2\pi f\varphi)$ are constants we obtain according to the elementary theorems on the Hankel transform [11]:

$$\begin{aligned}\mathscr{H}[\sin[2\pi f(r+\varphi)]] &= \mathscr{H}[\cos(2\pi f\varphi)\sin(2\pi fr)\\ &\quad+\sin(2\pi f\varphi)\cos(2\pi fr)]\\ &= \cos(2\pi f\varphi)\mathscr{H}[\sin(2\pi fr)]\\ &\quad+\sin(2\pi f\varphi)\mathscr{H}[\cos(2\pi fr)],\end{aligned}$$

so that

$$\begin{aligned}&\sin[2\pi f(r+\varphi)]\\ &\overset{\mathscr{H}}{\leftrightarrow} -\frac{f}{2\pi}\left[\cos(2\pi f\varphi)\frac{1}{(q^2-f^2)^{3/2}}\,\mathrm{step}\left(\frac{q}{2f}\right)\right.\\ &\left.\quad+\sin(2\pi\varphi)\frac{1}{(f^2-q^2)^{3/2}}\,\mathrm{rect}\left(\frac{q}{2f}\right)\right]\end{aligned}$$

or in terms of $\delta^{(1/2)}(x)$:

$$\begin{aligned}&\sin[2\pi f(r+\varphi)]\\ &\overset{\mathscr{H}}{\leftrightarrow} \frac{f}{\sqrt{\pi}}\frac{1}{(f+q)^{3/2}}\left[\cos(2\pi f\varphi)\delta^{(1/2)}(q-f)\right.\\ &\left.\quad+\sin(2\pi f\varphi)\delta^{(1/2)}(f-q)\right].\end{aligned} \tag{21}$$

We see, therefore, that the Fourier spectrum of the phase-shifted circular sine $\sin[2\pi f(r+\varphi)]$ is a circular impulse ring whose radial profile is a weighted sum of the spectra of the circular functions $\sin(2\pi fr)$ and $\cos(2\pi fr)$. Fig. 6, which has been obtained by 2D DFT, shows the spectra of $\sin[2\pi f(r+\varphi)]$ for various values of φ within the one-period range of $\varphi=0,\ldots,1/f$. As shown in Fig. 6, these spectra consist of a dipole- or quadrupole-like [12] impulse ring on the perimeter of a circle of radius f, which is surrounded in general by two wakes: one wake which trails off outwards (which is contributed by the spectrum of $\sin(2\pi fr)$), and a second wake which trails off towards the centre (which is contributed by the spectrum of $\cos(2\pi fr)$).

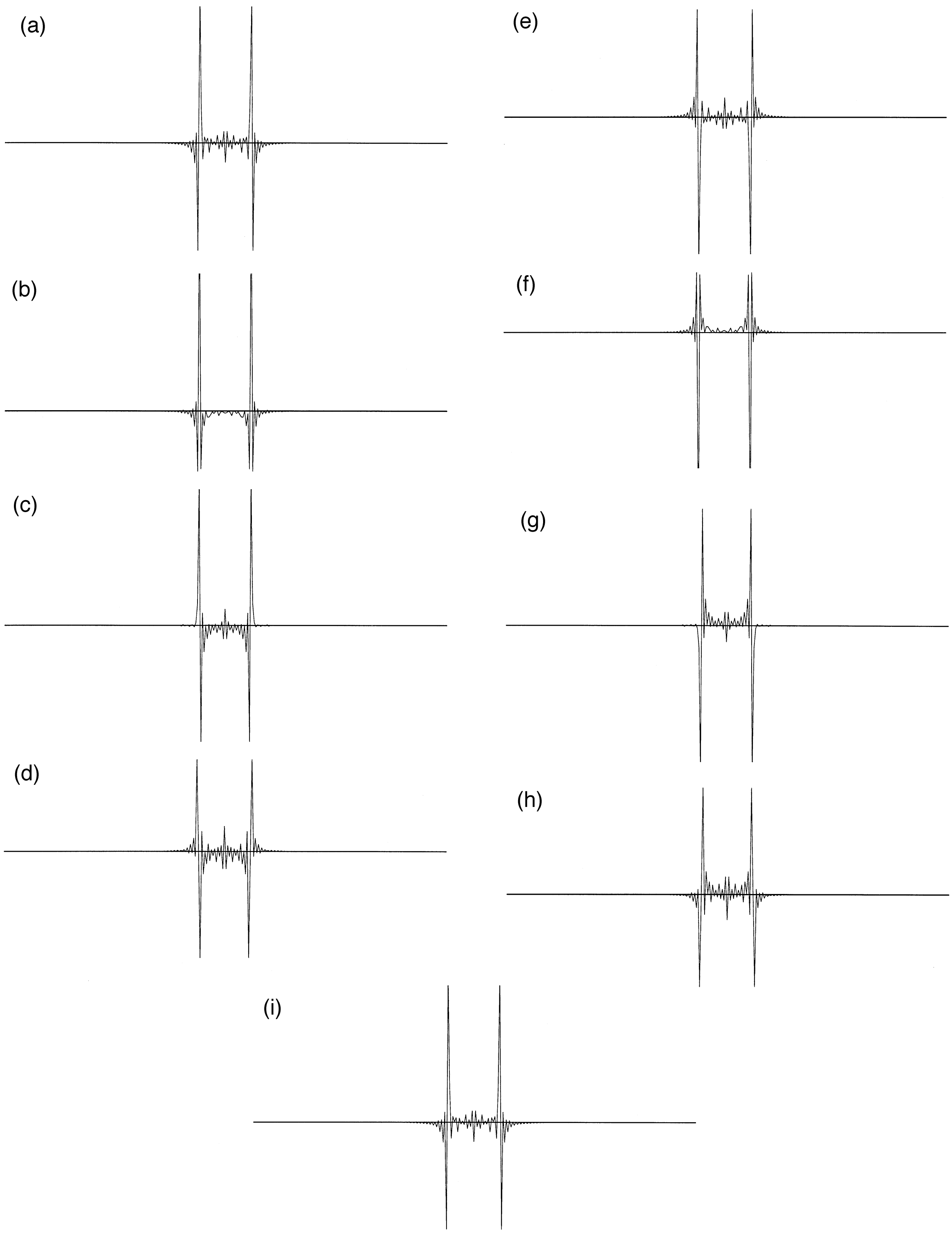

Fig. 6. One full evolution cycle of the spectrum of $\sin[2\pi f(r+\varphi)]$, $\varphi = 0, \ldots, 1/f$, illustrated by the average cross sections through the centre of the two dimensional DFT of $\sin[2\pi f(r+\varphi)]$ (averaged through all directions $\theta = 0°, \ldots, 360°$). (a) $\varphi = 0$; (b) $\varphi = \frac{1}{8f}$; (c) $\varphi = \frac{1}{4f}$; (d) $\varphi = \frac{3}{8f}$; (e) $\varphi = \frac{1}{2f}$; (f) $\varphi = \frac{5}{8f}$; (g) $\varphi = \frac{3}{4f}$; (h) $\varphi = \frac{7}{8f}$; (i) $\varphi = \frac{1}{f}$. Note that in each case the spectrum is a weighted sum of the spectra of $\sin(2\pi fr)$ (Fig. 1c) and $\cos(2\pi fr)$ (Fig. 2c); the corresponding weights are given in Table 1.

Table 1
The weighting coefficients $\sin(2\pi f\varphi)$ and $\cos(2\pi f\varphi)$ for the values of φ which appear in Fig. 6

φ	0	$\frac{1}{8f}$	$\frac{1}{4f}$	$\frac{3}{8f}$	$\frac{1}{2f}$	$\frac{5}{8f}$	$\frac{3}{4f}$	$\frac{7}{8f}$	$\frac{1}{f}$
$2\pi f\varphi$	0	$\frac{\pi}{4}$	$\frac{\pi}{2}$	$\frac{3\pi}{4}$	π	$\frac{5\pi}{4}$	$\frac{3\pi}{2}$	$\frac{7\pi}{4}$	2π
$\sin(2\pi f\varphi)$	0	$\frac{1}{2}\sqrt{2}$	1	$\frac{1}{2}\sqrt{2}$	0	$-\frac{1}{2}\sqrt{2}$	-1	$-\frac{1}{2}\sqrt{2}$	0
$\cos(2\pi f\varphi)$	1	$\frac{1}{2}\sqrt{2}$	0	$-\frac{1}{2}\sqrt{2}$	-1	$-\frac{1}{2}\sqrt{2}$	0	$\frac{1}{2}\sqrt{2}$	1

Similarly, using the trigonometric identity $\cos(\alpha+\beta) = \cos\alpha\cos\beta - \sin\alpha\sin\beta$ we obtain that the Fourier spectrum of the phase-shifted circular cosine function is given by:

$$\cos[2\pi f(r+\varphi)] \overset{\mathscr{H}}{\leftrightarrow} -\frac{f}{2\pi}\left[\cos(2\pi f\varphi)\frac{1}{(f^2-q^2)^{3/2}}\operatorname{rect}\left(\frac{q}{2f}\right) - \sin(2\pi f\varphi)\frac{1}{(q^2-f^2)^{3/2}}\operatorname{step}\left(\frac{q}{2f}\right)\right]$$

or in terms of $\delta^{(1/2)}(x)$:

$$\cos[2\pi f(r+\varphi)] \overset{\mathscr{H}}{\leftrightarrow} \frac{f}{\sqrt{\pi}}\frac{1}{(f+q)^{3/2}}\left[\cos(2\pi f\varphi)\delta^{(1/2)}(f-q) - \sin(2\pi f\varphi)\delta^{(1/2)}(q-f)\right]. \quad (22)$$

Let us now see in detail how the spectrum (21) is influenced when the constant radial phase-shift φ gradually varies throughout one full radial period of the sinusoidal function $\sin[2\pi f(r+\varphi)]$, say, between 0 and $T = 1/f$.

When $\varphi = 0$ the coefficients $\cos(2\pi f\varphi)$ and $\sin(2\pi f\varphi)$ are, respectively, 1 and 0, and therefore we obtain, in accordance with Eq. (18) (see Fig. 6a):

$$\sin(2\pi fr) \overset{\mathscr{H}}{\leftrightarrow} \frac{f}{\sqrt{\pi}}\frac{1}{(f+q)^{3/2}}\delta^{(1/2)}(q-f).$$

Now, when φ starts increasing the coefficient $\cos(2\pi f\varphi)$ starts to decrease, while the coefficient $\sin(2\pi f\varphi)$ starts to increase. This means that in the spectrum the weight of the outwards-oriented impulse ring $\delta^{(1/2)}(q-f)$ starts to decrease, while the weight of the inwards-oriented impulse ring $\delta^{(1/2)}(f-q)$ starts to increase. The two coefficients reach the point of equality when $\varphi = 1/8f$, where $\cos(2\pi f\varphi) = \sin(2\pi f\varphi) = \sqrt{2}/2$ (see Fig. 6b), and then they continue their respective increase (decrease) until $\varphi = 1/4f$, where $\cos(2\pi f\varphi) = 0$ and $\sin(2\pi f\varphi) = 1$. At this point $\sin[2\pi f(r+\varphi)]$ equals $\cos(2\pi fr)$, and indeed we obtain, in accordance with Eq. (11) (see Fig. 6c):

$$\sin\left[2\pi f\left(r+\frac{1}{4f}\right)\right] = \cos(2\pi fr) \overset{\mathscr{H}}{\leftrightarrow} \frac{f}{\sqrt{\pi}}\frac{1}{(f+q)^{3/2}}\delta^{(1/2)}(f-q).$$

Fig. 6 illustrates one full cycle of evolution for the spectrum of $\sin[2\pi f(r+\varphi)]$, where the radial phase φ varies throughout the full period between $\varphi = 0, \ldots, 1/f$. The coefficients $\sin(2\pi f\varphi)$ and $\cos(2\pi f\varphi)$ for the corresponding values of φ are given in Table 1. Note that the same figure also illustrates one full cycle of evolution for the spectrum of $\cos[2\pi f(r+\varphi)]$, where the radial phase φ varies throughout a full period between $\varphi = -1/4f, \ldots, 3/4f$.

5. Conclusions

The Fourier transforms of the circular sine and cosine functions do not appear in standard tables of Fourier (or Hankel) transform pairs, since these functions do not properly decay and consequently their Fourier transforms include a ‘wild’ (impulsive) behaviour on the border of their circular spectrum support. We have shown that this impulsive behaviour can be expressed mathematically in terms of the half-order derivative of the impulse ring $\delta(q-f)$. The spectrum of the circular sine function $\sin(2\pi fr)$ consists of a circular ring of radius f with a positive impulsive behaviour in its internal side, and a negative impulsive behaviour in its external side which gradually trails off outwards in the form of a negative, continuous wake. The spectrum of the circular cosine function $\cos(2\pi fr)$, on its way, consists of a circular ring of radius f with a positive impulsive behaviour in its outer side, and a negative impulsive behaviour in its inner side which gradually trails off towards the spectrum centre in the form of a negative, continuous wake. In fact, the spectrum of the circular cosine can be seen as an inside-out inversion of the spectrum of the circular sine; this is so due to the inside-out inversion of the half-order derivative of the impulse ring in the spectrum of the circular cosine (compare Eqs. (11) and (18)).

Generalizing these results to the most general form of circular sinusoidal functions, namely: circular sine or cosine functions with any arbitrary phase, we have shown that their spectra are obtained as a weighted sum of the spectra of the pure circular sine and cosine functions. In general, their spectra consist, therefore, of an impulse ring of radius f which has some peculiar properties, notably a wake which trails off outwards (the weighted contribution of the spectrum of $\sin(2\pi fr)$), and a second wake which trails off towards the centre (the weighted contribution of the spectrum of $\cos(2\pi fr)$).

Finally, we have shown that the transitions undergone by the spectrum as the radial phase is being varied throughout one full period are continuous and gradual.

Appendix A. A further verification

It is interesting to note that the Hankel transforms of $\cos(2\pi fr)$ and $\sin(2\pi fr)$ (Eqs. (5) and (16)) can be also verified in an independent way by means of the identity:

$$\exp(-\mathrm{i}2\pi fr) = \cos(2\pi fr) - \mathrm{i}\sin(2\pi fr).$$

According to Ref. [13] the Hankel transform of $\exp(-ar)$ is given by:

$$\exp(-ar) \overset{\mathscr{H}}{\leftrightarrow} \frac{2\pi a}{(4\pi^2 q^2 + a^2)^{3/2}}$$

and therefore by taking $a = \mathrm{i}2\pi f$ we obtain:

$$\exp(-\mathrm{i}2\pi fr) \overset{\mathscr{H}}{\leftrightarrow} \frac{\mathrm{i}f}{2\pi} \frac{1}{(q^2 - f^2)^{3/2}}. \qquad \text{(A.1)}$$

Now, using Eqs. (5) and (16) we obtain:

$$\cos(2\pi fr) - \mathrm{i}\sin(2\pi fr)$$

$$\overset{\mathscr{H}}{\leftrightarrow} -\frac{f}{2\pi}\left[\frac{1}{(f^2-q^2)^{3/2}}\,\mathrm{rect}\left(\frac{q}{2f}\right) - \mathrm{i}\frac{1}{(q^2-f^2)^{3/2}}\,\mathrm{step}\left(\frac{q}{2f}\right)\right]$$

$$= \frac{\mathrm{i}f}{2\pi}\left[\frac{1}{(q^2-f^2)^{3/2}}\,\mathrm{step}\left(\frac{q}{2f}\right) - \frac{1}{\mathrm{i}(f^2-q^2)^{3/2}}\,\mathrm{rect}\left(\frac{q}{2f}\right)\right]$$

and since $-\mathrm{i} = \mathrm{i}^3 = (-1)^{3/2}$:

$$= \frac{\mathrm{i}f}{2\pi}\left[\frac{1}{(q^2-f^2)^{3/2}}\,\mathrm{step}\left(\frac{q}{2f}\right) + \frac{1}{(q^2-f^2)^{3/2}}\,\mathrm{rect}\left(\frac{q}{2f}\right)\right]$$

$$= \frac{\mathrm{i}f}{2\pi}\frac{1}{(q^2-f^2)^{3/2}}$$

and this is, indeed, identical to the right hand side of Eq. (A.1).

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