A Contribution to 3D Digital Lines.

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Abstract

We propose in this paper a new approach to three-dimensional digital lines (3DDLs) based on the study of the integer lattice generated by the projection of $\mathbb{Z}^3$ onto an euclidean plane which reduces the problem to dimension 2. The many properties of this lattice lead to an arithmetical definition of 3DDLs in accordance with a topological characterization. This definition is then used in an algorithm that calculates the intersection between a naive 3DDL and an arbitrary digital plane. We also show that this algorithm can be extended to calculate the intersection between a plane and a set of adjacent 3DDLs incrementally in a very efficient manner.

1 Investigating a definition of 3D digital lines

1.1 Introduction

Finding the closest integer points to a given integer direction is a question that is solved in dimension 2 through Euclid's and related continued fraction algorithms [HW79] and a theorem of Klein [Rev91]. However such a problem has not been efficiently solved in a three-dimensional space. Most of the works on the subject are attempts to generalize Euclid's and continued fraction algorithms to higher dimensions [Jus92, Ros42]. In fact this theoretical problem is closely related to computer graphics and more especifically to the definition and properties of three-dimensional digital lines (3DDLs). The study of 3DDLs can thus take great advantage of the results obtained in arithmetics and that is what guided this research.

More precisely our goal is to find a definition for a line of direction $(a, b, c) \in \mathbb{Z}^3$ in a discrete 3-dimensional space and to examine its properties. Our original approach is to consider the projection of the discrete space $\mathbb{Z}^3$ along this direction onto an orthogonal plane. Such a projection yields a rational lattice of points the study of which leads to many results partially described in this paper. Among these is a convenient arithmetical definition of 3D digital lines.

1.2 A topological definition of 3D digital lines.

Two-dimensional digital lines are generally defined by the digitization of continuous lines which can be done in several ways [CM91]. This definition has been developed into more synthetic representations of 2D digital lines [DS84]. A more flexible arithmetical definition has also been proposed [Rev91], which allows the representation of standard 8-connected lines (which are called *naive*) as well as thicker and thiner lines and thus leads to a much more general theory of these objects. However such a theory does not exist for 3DDLs and most people working on computer graphics restrict their point of view on 3DDLs to the simple digitization of their continuous equivalent.

As a starting point we propose to define topologically the set that we will call a *naive 3D digital line* :

Definition 1.1 *A* naive three-dimensional digital line *is a subset of* $\mathbb{Z}^3$ *defined by the intersection of two digital planes and verifying the following conditions :*

- *it is* 26-connected,

- *it is* minimal *in the sense that removing any element splits the subset into two separate 26-connected components.*

We will show that for any rational direction there exists at least one such subset and that the discretization of a three-dimensional continuous line is a naive 3DDL according to this definition.

1.3 Projection of $\mathbb{Z}^3$ onto an euclidean plane

Since 3DDLs are subsets of $\mathbb{Z}^3$, it is natural to try to ease their study by reducing the problem to two dimensions. We obtain this reduction by projecting onto a plane orthogonal to the direction of the 3DDL. As we will see the properties of this projection define accurately the line in the three-dimensional space.

For convenience in the mathematics we will only consider directions (a, b, c) such as $0 \leq a \leq b \leq c$. This is not properly a restriction since the problem has inherent symmetries which allow a straightforward generalization to the other directions. Moreover, in order to simplify some calculations we also impose that (a, b, c) should be mutually prime. Then we define (u, v) and (u', v') :

$$au + cv = 1 \quad bu' + cv' = 1 \tag{1}$$

In addition let $\omega^2 = a^2 + b^2 + c^2$ and $\mathcal{P}$ be the euclidean plane, $\mathcal{P} \ : \ ax + by + cz = 0$. We denote $\pi()$ the projection operator on $\mathcal{P}$ along the direction (a, b, c).

A point $A(x, y, z) \in \mathbb{Z}^3$ is projected along the direction (a, b, c) onto a point $A'(x', y', z')$ of the plane $\mathcal{P}$ through the relation :

$$A' = \pi(A) = \mathcal{M}A \quad \text{where} \quad \mathcal{M} = \frac{1}{\omega^2}\begin{pmatrix} b^2 + c^2 & -ab & -ac \\ -ab & a^2 + c^2 & -bc \\ -ac & -bc & a^2 + b^2 \end{pmatrix} \tag{2}$$

We project $\mathbb{Z}^3$ as a whole using this relation. This yields a rational lattice $\mathcal{L}$ on the plane $\mathcal{P}$. We also call $\mathcal{L}_s$ the *sparse lattice* generated on $\mathcal{P}$ by the projection of the integer points of the main plane xOy and we call $\mathcal{L}_d$ (for *dense lattice*) the result of the scaling by $1/c$ of $\mathcal{L}_s$. The integer points of xOy being part of $\mathbb{Z}^3$ it is obvious that :

$$\mathcal{L}_s \subset \mathcal{L}$$

and by definition we also have $\mathcal{L}_s \subset \mathcal{L}_d$, we will show later that $\mathcal{L} \subset \mathcal{L}_d$ and thus we have the global relation (see Figure 1) :

$$\mathcal{L}_s \subset \mathcal{L} \subset \mathcal{L}_d$$

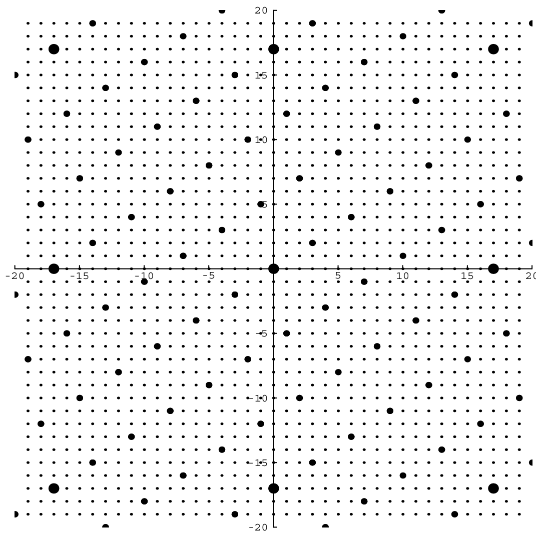

Figure 1: The lattices $\mathcal{L}_d$, $\mathcal{L}$ and $\mathcal{L}_s$ associated to the projection of $\mathbb{Z}^3$ on $\mathcal{P}(5{,}9{,}17)$ seen in basis $(\vec{\imath}, \vec{\jmath})$.

Let $(\vec{I}, \vec{J}, \vec{K})$ be the fundamental basis of $\mathbb{Z}^3$. A basis $(\vec{\imath_s}, \vec{\jmath_s})$ of $\mathcal{L}_s$ is given by the projection of the two vectors $(-\vec{I}, -\vec{J})$ which is described by the first two columns of $\mathcal{M}$. Then, a basis $(\vec{\imath}, \vec{\jmath})$ of $\mathcal{L}_d$ is obtained by scaling this basis by $1/c$:

$$\vec{\imath} = \frac{1}{c\omega^2}\begin{pmatrix} -(b^2+c^2) \\ ab \\ ac \end{pmatrix} \qquad \vec{\jmath} = \frac{1}{c\omega^2}\begin{pmatrix} ab \\ -(a^2+c^2) \\ bc \end{pmatrix} \tag{3}$$

We can prove that this basis is also a basis of $\mathcal{L}$ thus showing that $\mathcal{L} \subset \mathcal{L}_d$

by solving the following equation :

$$\forall(x,y,z) \in \mathcal{L}, \quad \exists(k,l) \in \mathbb{Z}^2 \quad / \;\mathcal{M}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = k\vec{\imath} + l\vec{\jmath}$$

which gives :

$$\left\{\begin{array}{l} k = -cx + az \\ l = -cy + bz \end{array}\right. \quad \text{and} \quad \left\{\begin{array}{l} x = \left\{\frac{-kv}{a}\right\} + \lambda a \\ y = \left\{\frac{-lv'}{b}\right\} + \lambda b \\ z = \left\{\frac{ku}{c}\right\} + \lambda c = \left\{\frac{lu'}{c}\right\} + \lambda c \qquad \lambda \in \mathbb{Z} \end{array}\right. \tag{4}$$

Notation The square brackets $\left[\frac{m}{n}\right]$, and the curly brackets $\left\{\frac{m}{n}\right\}$ denote respectively the quotient and the remainder of the euclidean division of m by n.

A simple parameterization of the lattice $\mathcal{L}$ in the basis $(\vec{\imath}, \vec{\jmath})$ is deduced from the expression of (k,l) in Equation 4 :

$$\mathcal{L} = \left\{ x\begin{pmatrix} c \\ 0 \end{pmatrix} + y\begin{pmatrix} 0 \\ c \end{pmatrix} + z\begin{pmatrix} a \\ b \end{pmatrix} / \forall(x,y,z) \in \mathbb{Z}^3 \right\} \tag{5}$$

A proper linear combination of the first and third vectors generates a minimum shift along the first axis :

$$v\begin{pmatrix} c \\ 0 \end{pmatrix} + u\begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} 1 \\ bu \end{pmatrix}$$

Therefore the couple of vectors $\left(\begin{pmatrix} 0 \\ c \end{pmatrix}, \begin{pmatrix} 1 \\ bu \end{pmatrix}\right)$ is a basis of the lattice $\mathcal{L}$.

We call *lattice of remainders* of the fraction (or slope) $\frac{a}{b}$, the integer lattice $\mathcal{R}\left(\frac{a}{b}\right)$ obtained by repeating periodically along the two directions the pattern defined by :

$$\left(i, \left\{\frac{ai}{b}\right\}\right) \qquad i \in \mathbb{Z}$$

In the same way we also denote $\mathcal{R}^{\perp}\left(\frac{a}{b}\right)$ the integer lattice defined by the repetition along the two directions of the pattern defined by :

$$\left(\left\{\frac{ai}{b}\right\}, i\right) \qquad i \in \mathbb{Z}$$

Thus we have :

Proposition 1.1 *The projection of $\mathbb{Z}^3$ along the direction (a, b, c) onto the euclidean plane $\mathcal{P}$ associated to the basis $(\vec{\imath}, \vec{\jmath})$ yields an integer lattice $\mathcal{L}$ which coincides with the lattice of remainders $\mathcal{R}\left(\frac{bu}{c}\right)$ and $\mathcal{R}^{\perp}\left(\frac{au'}{c}\right)$.*

This result is fundamental since it offers a very simple parameterization of $\mathcal{L}$ and thus an efficient way to walk over the points of this lattice.

1.4 Characterization of digital lines of $\mathbb{Z}^3$

Since we have considered the projection on a plane of normal direction (a, b, c) it is natural to look for the projection of a naive 3DDL of direction (a, b, c) as the subset of $\mathcal{L}$ of smallest area having for inverse image a connected path in $\mathbb{Z}^3$. Let us then consider the points of $\mathcal{L}$ such as :

$$0 \leq k < c \quad \text{and} \quad 0 \leq l < c \tag{6}$$

According to Proposition 1.1, these points correspond to the set

$$\left\{\left(i, \left\{\frac{bui}{c}\right\}\right) / \, 0 \leq i < c\right\} \tag{7}$$

Then, Equation 4 maps these points to the following ones :

$$\begin{cases} x = \left\{\frac{-vi}{a}\right\} + \lambda a \\ y = \left\{\frac{\left\lfloor\frac{bui}{c}\right\rfloor}{b}\right\} + \lambda b \\ z = \left\{\frac{ui}{c}\right\} + \lambda c & \qquad \lambda \in \mathbb{Z} \end{cases}$$

A simple calculation shows that this set is equivalent to the one defined by :

$$\begin{cases} x = \left\lfloor\frac{ai}{c}\right\rfloor \\ y = \left\lfloor\frac{bi}{c}\right\rfloor \\ z = i & \qquad i \in \mathbb{Z} \end{cases}$$

which corresponds to the discretization by truncation of the euclidean line $\left\{ M \in \mathbb{R}^3 / \overrightarrow{OM} = t(a\ b\ c), t \in \mathbb{R} \right\}$. It is easy to show that this set is 26-connected. Furthermore Equation 6 shows that it is equivalent to the intersection of two digital planes :

$$\begin{cases} \mu \leq & -cx & & +az & < \mu + c \\ \mu' \leq & & -cy & +bz & < \mu' + c \end{cases}$$

Therefore this set defines a *naive 3D digital line.*

Thus with the conditions $(a, b, c) \in \mathbb{Z}^3$ and $0 < a < b < c$ we can write :

Proposition 1.2 *Let S be the subset of the squares of $\mathcal{P}$(a,b,c) of edges parallel to $\vec{\imath}$ or $\vec{\jmath}$ and of edge length c, then the inverse image in $\mathbb{Z}^3$ of any square of S under the orthogonal projection onto $\mathcal{P}$ is a* naive 3D digital line *of direction (a,b,c).*

Proposition 1.3 *The subset of $\mathbb{Z}^3$ defined by the following system :*

$$\begin{cases} \mu \leq & -cx & & +az & < \mu + c \\ \mu' \leq & & -cy & +bz & < \mu' + c \end{cases}$$

is a naive 3D digital line *of direction (a,b,c).*

Proposition 1.4 *The set of integer points contained in a cylinder of axis (a,b,c) intersecting the base plane (xOy) on a unit square is a* naive 3D digital line.

The first two propositions were established before and the last one derives directly from the former by considering z=0 in the equations.

2 Intersection of digital lines and planes

As shown on Figure 2 and due to the discrete nature of these objects, the intersection between digital lines and planes is often a rather complex set consisting of many voxels and the structure of which is not apparent. Our goal in this part is to write an efficient walking algorithm for this kind of intersection.

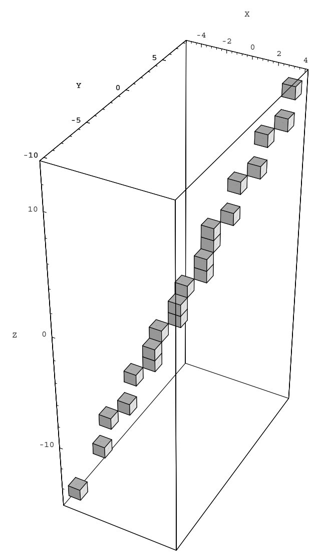

Figure 2: The intersection of a naive 3DDL $\mathcal{D}$(7,15,23) and naive digital plane $\mathcal{P}$(-8,29,-15,-25).

2.1 Preliminary definitions and restrictions

We use here the definition of digital planes given by Reveillès in [DRR94]. A digital plane with (d, e, f) normal vector is the set of integer points (x, y, z) satisfying a double diophantine inequality :

$$\mathcal{P}(d, e, f, \gamma, \rho) \ : \ \gamma \leq dx + ey + fz < \gamma + \rho \tag{8}$$

where *all parameters are integers.*

We also consider a 3DDL under the form given in Proposition 1.3 :

$$\mathcal{D}(a,b,c)\ :\left\{\begin{array}{lccll}\mu \leq & -cx & & +az & < \mu+\omega \\ \mu' \leq & & -cy & +bz & < \mu'+\omega'\end{array}\right. \tag{9}$$

$$\text{where } \omega=\omega'=c \qquad (0<a<b<c) \tag{10}$$

Thus, the intersection of a naive 3DDL with a digital plane is in fact the intersection of three digital planes which is similar to the equivalent continuous problem.

For the clarity of the presentation we will consider the digital lines of the standard simplex $(0 < a < b < c)$ only and any kind of plane (not necessarily naive and of any direction). Thanks to the group of symmetries of the cube we will thus cover the general case.

2.2 Mathematical determination of the intersection

The following matricial system defines the intersection :

$$\begin{pmatrix}\mu\\ \mu\\ \gamma\end{pmatrix} \leq \begin{pmatrix}-c & 0 & a\\ 0 & -c & b\\ d & e & f\end{pmatrix}\begin{pmatrix}x\\ y\\ z\end{pmatrix} < \begin{pmatrix}\mu+c\\ \mu+c\\ \gamma+\rho\end{pmatrix} \tag{11}$$

Of course this system must be solved in $\mathbb{Z}^3$. The key idea is to find an appropriate unimodular matrix $\mathcal{U}$ (ie of determinant equal to 1) defining a bijective transform that turns the system easier to analyze.

After some calculation we are led to :

$$\mathcal{U}=\begin{pmatrix}-v-and' & 0 & a\\ d'm-bnd' & -1 & b\\ u-cnd' & 0 & c\end{pmatrix} \quad \text{where} \quad \left\{\begin{array}{l}d'=-dv+fu\\ e'=-e\\ f'=ad+be+cf\end{array}\right. \tag{12}$$

with the additional condition that e' and f' be relatively prime which ensures that there exist (m,n) such as $me'+nf'=1$.

System 11 is reduced to the following form :

$$\begin{pmatrix}\mu\\ \mu\\ \gamma\end{pmatrix} \leq \begin{pmatrix}1 & 0 & 0\\ bu-md'c & c & 0\\ 0 & e' & f'\end{pmatrix}\begin{pmatrix}X\\ Y\\ Z\end{pmatrix} < \begin{pmatrix}\mu+c\\ \mu+c\\ \gamma+\rho\end{pmatrix} \tag{13}$$

with $(x\ y\ z) = \mathcal{U}.(X\ Y\ Z)$

This equation can be rewritten in the following algorithmic way :

$$\begin{cases} \mu \le X < \mu + c \\ Y = md'X - \left[\frac{buX-\mu}{c}\right] \\ -\left[\frac{e'Y-\gamma}{f'}\right] \le Z < -\left[\frac{e'Y-\gamma-\rho}{f'}\right] \end{cases}$$

This shows that :

X steps through c values

for each value of X there is a unique value of Y

for each value of Y there are $N(Y)$ values for Z :

$N(Y) = -\left[\frac{-\rho}{f'}\right]$ if $\left\{\frac{e'Y-\gamma}{f'}\right\} < f' - \left\{\frac{-\rho}{f'}\right\}$

$N(Y) = -\left[\frac{-\rho}{f'}\right] - 1$ if $\left\{\frac{e'Y-\gamma}{f'}\right\} \ge f' - \left\{\frac{-\rho}{f'}\right\}$

Proposition 2.1 *The intersection of a naive digital line $\mathcal{D}(a,b,c,\mu)$ and a digital plane $\mathcal{P}(d,e,f,\gamma,\rho)$ consists of N voxels with*

$$-c\left[\frac{-\rho}{f'}\right] - c \le N \le -c\left[\frac{-\rho}{f'}\right]$$

2.3 Incremental aspect of the digital line-plane intersection

The simple previous algorithm has nice properties when it comes to compute the intersection between a digital plane and successive adjacent 3DDLs. We will carry the demonstration for a set of naive parallel digital lines adjacent along the x-axis. The problem being perfectly symmetrical, the results will be easily transposed to adjacency on the other two axes.

We derive the arithmetical definition for the a line of direction (a,b,c) going through the point (x_0, y_0, z_0) form Equation 9 :

$$\begin{cases} -cx_0 + az_0 \le & -cx & & +az & < -cx_0 + az_0 + c \\ -cy_0 + bz_0 \le & & -cy & +bz & < -cy_0 + bz_0 + c \end{cases} \tag{14}$$

In what follows we denote the following useful value as δ :

$$\delta = \left\{\frac{e'Y-\gamma}{f'}\right\} \tag{15}$$

Let us consider the digital line going through the voxel (x_0+1, y_0, z_0) and let us examine how the different parameters used in the computation of the intersection with the plane $\mathcal{P}(d, e, f, \gamma, \rho)$ evolve in regards to their value for the line going through (x_0, y_0, z_0).

Notation : We use the notation P for a parameter related to the line going through voxel (x_0, y_0, z_0) and P'_x for the same parameter but related to the line going through (x_0+1, y_0, z_0) (unit shift along the x-axis).

After some calculation the following results can be established :

$$\boxed{\begin{aligned}
x'_x &= x + 1 - a\left[\frac{d}{f'}\right] - a\left[\frac{\delta+\left\{\frac{d}{f'}\right\}}{f'}\right]\\
y'_x &= y - b\left[\frac{d}{f'}\right] - b\left[\frac{\delta+\left\{\frac{d}{f'}\right\}}{f'}\right]\\
z'_x &= z - c\left[\frac{d}{f'}\right] - c\left[\frac{\delta+\left\{\frac{d}{f'}\right\}}{f'}\right]\\
\delta'_x &= \delta + \left\{\frac{d}{f'}\right\} - f'\left[\frac{\delta+\left\{\frac{d}{f'}\right\}}{f'}\right]\\
N(Y)'_x &= -\left[\frac{-\rho}{f'}\right] - \left[\frac{\delta'+\left\{\frac{d}{f'}\right\}}{f'}\right]
\end{aligned}}$$

Table 1: Evolution of intersection parameters for X-contiguous lines.

From these equations it is clear that once the process is initialized for the first line, there is very little computation to determine the intersections of the following lines. In particular there are only add/subtract and test operations involved.

3 Conclusion

We have shown that 3DDLs can be studied through an original approach consisting in projecting them onto a real continuous plane orthogonally to their direction. It has led us to an arithmetical representation similar to the definition of 2DDLs and digital planes proposed by Reveillès [Rev91]. Thanks to this definition a walking algorithm for the intersection of 3DDLs and digital planes has been written. Boundaries for the number of voxels in these intersections have also been found that depend only on the intrinsic parameters of the intersecting objects. This approach promises more intersting results that are currently under study.

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