Extension of the Williams–Clapper model to stacked nondiffusing colored coatings with different refractive indices

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We propose a model for predicting the reflectance and transmittance of multiple stacked nonscattering coloring layers that have different refractive indices. The model relies on the modeling of the reflectance and transmittance of a bounded coloring layer, i.e., a coloring layer and its two interfaces with neighboring media of different refractive indices. This model is then applied to deduce the reflectance of stacked nonscattering layers of different refractive indices superposed with a reflecting diffusing background that has its own refractive index. The classical Williams–Clapper model becomes a special case of the proposed stacked layer model. © 2006 Optical Society of America

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1. INTRODUCTION

Predicting the reflectance factor of multilayer samples such as printed or painted samples requires modeling of the interaction of light with a medium made of several superposed layers that have different spectral absorbances, light scattering properties, and possibly different refractive indices. Two classical models enable prediction of spectral reflectance factors: the Kubelka–Munk model,[1] and the Williams–Clapper model.[2] The Kubelka–Munk model and its extensions[3,4] are restricted to diffusing layers all having the same index of refraction. The Williams–Clapper model is restricted to a single coloring nondiffusing layer superposed on top of a diffusing layer of the same refractive index.

In the present contribution, we propose a model predicting the reflectance of a diffusing support coated with nondiffusing coloring layers of different refractive indices. Fresnel reflections and refractions occur at the interfaces between neighboring layers. In each of the nondiffusing layers, pencils of light propagate along straight lines, are absorbed according to Beer's law, and are internally reflected or refracted into an adjacent layer in a direction given by Snell's laws and with a proportion given by the Fresnel formulas. Beneath the coloring layers, the diffusing support reflects toward the coloring layers a Lambertian light composed of an infinity of pencils of light that have uniformly distributed orientations. Neither the Kubelka–Munk model nor the Williams–Clapper model is able to predict the reflectance of such a multilayer sample.

The Kubelka–Munk model[1] predicts the reflectance and transmittance of a diffusing and absorbing layer for light propagating according to the layer's normal direction. This model has been extended by Saunderson to account for the internal reflections at the interface between the scattering medium and the air,[5] by Kubelka to account for diffuse light,[3] and again by Kubelka[4] to account for multiple superposed layers all having the same index of refraction.

The Williams–Clapper model,[2] in contrast to the Kubelka–Munk model, requires the color layer to be nondiffusing. In the original model, the illuminating light is collimated and incident at an angle of 45° and the exiting light is captured by a radiance detector at a zero angle. Extensions have been proposed for any measuring geometry.[6–8] The Williams–Clapper model computes for each pencil of the diffuse light reflected by the reflecting diffuse substrate its absorption according to Beer's law and its reflectance at the interface with the air due to the Fresnel reflectivity. The attenuation of diffuse light due to absorption in the coloring layer and reflection at the interface with the air is thus obtained by summing up the attenuation of all light pencils composing the diffuse light. However, the Williams–Clapper model is limited to a color layer of the same refractive index as the underlying reflecting diffuse substrate.

Since the Kubelka–Munk model is not applicable to nonscattering layers, we pursue the Williams–Clapper approach in order to predict the reflectance of a diffusing support superposed with several nondiffusing coloring layers of different refractive indices. The resulting comprehensive multilayer reflectance model includes as special cases the Williams–Clapper model[2] as well as the air–paint[7] and the air–varnish–paint[9] reflection models.

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In Section 2, we recall the basics of light reflection and transmission at an interface both for collimated and for diffuse natural light. In Section 3, we first establish the reflectance and transmittance of a single nonscattering coloring layer bounded by two interfaces. By replacing the reflectance (respectively, transmittance) of a simple interface between two media with the reflectance (respectively, transmittance) of a bounded layer, we are able to deduce the reflectance and transmittance of multiple stacked nonscattering layers of different refractive indices. In Section 4, we compute the global reflectance and the bidirectional reflectance distribution function (BRDF) of stacked nonscattering layers superposed on top of a diffusing medium that has its own intrinsic reflectance and refractive index. The Williams–Clapper model becomes a special case of the proposed comprehensive multilayer reflectance prediction model. In Section 5, we draw the conclusions.

2. BASIC CONCEPTS: REFLECTION, TRANSMISSION, AND ABSORPTION

All irradiances, reflectances, transmittances, and absorption coefficients are wavelength dependent.

When a light pencil reaches an interface between two media j and k of different refractive indices n_j and n_k, one part is reflected and one part is transmitted (refracted). The reflection and refraction directions satisfy Snell's laws (Fig. 1):

$$n_j \sin \theta_j = n_k \sin \theta_k. \tag{1}$$

For the considered interfaces, within the visible wavelength range, we assume that the imaginary part of the refractive index is very small compared with the real part. Nevertheless, all expressions can be generalized to complex refractive indices.

The reflectivity and the transmittivity of the interface, i.e., the fraction of incident irradiance that is reflected and transmitted by the interface, are given by the Fresnel formulas. They depend on the polarization of the incident light[10] and may be expressed in terms of the reflectivity and transmittivity associated with polarization in both the parallel and the perpendicular directions with respect to the incidence plane. The reflectivity for the parallel component, denoted by superscript (p), is

$$R_{jk}^{(p)}(\theta_j) = \frac{\tan^2(\theta_j - \theta_k)}{\tan^2(\theta_j + \theta_k)}, \tag{2}$$

and the reflectivity for the perpendicular component, denoted by superscript (s), is

$$R_{jk}^{(s)}(\theta_j) = \frac{\sin^2(\theta_j - \theta_k)}{\sin^2(\theta_j + \theta_k)} \tag{3}$$

In this paper, we consider that the incident light is incoherent and unpolarized (natural light). The directions of vibration of natural light vary rapidly in a random manner. The wave component polarized parallel to the incidence plane, averaged over all the directions of vibration, forms an irradiance $W_i^{(p)}$. The average wave component polarized perpendicular to the plane of incidence

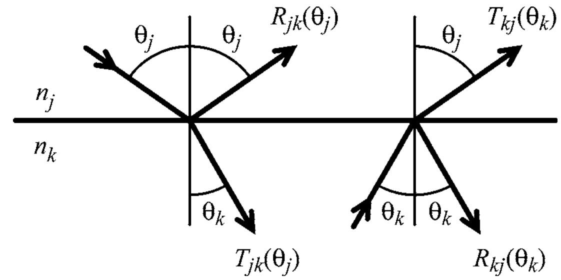

Fig. 1. Reflection and transmission of light at an interface between two media of refractive indices n_j and n_k.

forms an irradiance $W_i^{(s)}$. Since natural light is incoherent and unpolarized, the two linearly polarized irradiance components are independent and equal:

$$W_i^{(p)} = W_i^{(s)} = W_i/2. \tag{4}$$

The total incident irradiance W_i is the sum of the two linearly polarized irradiances:

$$W_i = W_i^{(p)} + W_i^{(s)}. \tag{5}$$

When the irradiance W_i is reflected by the interface, the reflected irradiance W_r is composed of a parallelly polarized component $W_r^{(p)} = R_{jk}^{(p)}(\theta_j) W_i^{(p)}$ and a perpendicularly polarized component $W_r^{(s)} = R_{jk}^{(s)}(\theta_j) W_i^{(s)}$. According to Eqs. (4) and (5), the Fresnel reflection coefficient $R_{jk}(\theta_j)$ for incident natural light is

$$R_{jk}(\theta_j) = \tfrac{1}{2}[R_{jk}^{(p)}(\theta_j) + R_{jk}^{(s)}(\theta_j)]. \tag{6}$$

The reflected light is partially polarized, since the two linearly polarized irradiances $W_r^{(p)}$ and $W_r^{(s)}$ are different due to the difference between the Fresnel reflection coefficients $R_{jk}^{(p)}(\theta_j)$ and $R_{jk}^{(s)}(\theta_j)$. The reflected light is still incoherent; i.e., the irradiances $W_r^{(p)}$ and $W_r^{(s)}$ are independent. If the reflected light reaches a new interface, the Fresnel coefficient $R_{jk}(\theta_j)$ expressed in Eq. (6) cannot be applied. However, we may consider separately the reflection of the irradiance $W_r^{(p)}$ and the reflection of the irradiance $W_r^{(s)}$ and sum the resulting reflected irradiances.

Since the energy is conserved at the interface, the Fresnel transmission coefficients $T_{jk}^{(p)}(\theta_j)$, $T_{jk}^{(s)}(\theta_j)$, and $T_{jk}(\theta_j)$ are related to the Fresnel reflection coefficients $R_{jk}^{(p)}(\theta_j)$, $R_{jk}^{(s)}(\theta_j)$, and $R_{jk}(\theta_j)$, respectively, by

$$T_{jk}^{*}(\theta_j) = 1 - R_{jk}^{*}(\theta_j). \tag{7}$$

Similarly to the reflected light, the transmitted light is also incoherent and partially polarized; i.e., the two transmitted linearly polarized irradiances are independent but not equal. Therefore, if the incident natural light interacts successively with several interfaces (reflections or transmissions), we have to consider separately the s-polarized component and the p-polarized component.

For both polarizations, and thereby for natural light, the reflection and transmission coefficients verify the following property,

$$R_{jk}^{*}(\theta_j) = R_{kj}^{*}(\theta_k), \tag{8}$$

and consequently

$$T^*_{jk}(\theta_j) = T^*_{kj}(\theta_k). \tag{9}$$

A light pencil passing from a medium m_j of refractive index n_j to a medium m_k of refractive index n_k is subject to refraction. According to Snell's laws, the refraction modifies the pencil's main direction θ_i into a direction θ_k and its solid angle $\mathrm{d}\Omega_i$ into a solid angle $\mathrm{d}\Omega_k$, according to the relation[11]

$$\mathrm{d}\Omega_k = \left(\frac{n_j}{n_k}\right)^2 \frac{\cos\theta_j}{\cos\theta_k}\mathrm{d}\Omega_j. \tag{10}$$

A. Diffuse Reflectance of an Interface

Let us consider an interface between media of different refractive indices n_j and n_k illuminated by a Lambertian irradiance W_i composed of natural light. The interface receives from all directions (θ_j, ϕ_j) of the upper hemisphere a constant radiance $L_i = W_i/\pi$, composed of a p-polarized component $L_i^{(p)}$ and an s-polarized component $L_i^{(s)}$:

$$L_i^{(p)} = L_i^{(s)} = \frac{L_i}{2} = \frac{W_i}{2\pi}. \tag{11}$$

Let us first consider the p-polarized component. The element of irradiance $\mathrm{d}W_i^{(p)}(\theta_j, \phi_j)$ received from a direction (θ_j, ϕ_j) is related to the radiance $L_i^{(p)}$ by

$$\mathrm{d}W_i^{(p)}(\theta_j, \phi_j) = L_i^{(p)} \cos\theta_j \mathrm{d}\Omega_j = L_i^{(p)} \cos\theta_j \sin\theta_j \mathrm{d}\theta_j \mathrm{d}\phi_j. \tag{12}$$

This element of irradiance $\mathrm{d}W_i^{(p)}(\theta_j, \phi_j)$ is reflected by the interface within a proportion $R_{jk}^{(p)}(\theta_j)$ given by the Fresnel formulas. The reflected element of irradiance $\mathrm{d}W_r^{(p)}(\theta_j, \phi_j)$ is therefore

$$\mathrm{d}W_r^{(p)}(\theta_j, \phi_j) = R_{jk}^{(p)}(\theta_j) L_i^{(p)} \cos\theta_j \sin\theta_j \mathrm{d}\theta_j \mathrm{d}\phi_j. \tag{13}$$

Similarly, the interface receiving the s-polarized radiance $L_i^{(s)}$ from the same direction from the same direction (θ_j, ϕ_j) reflects an element of irradiance $\mathrm{d}W_r^{(s)}(\theta_j, \phi_j)$:

$$\mathrm{d}W_r^{(s)}(\theta_j, \phi_j) = R_{jk}^{(s)}(\theta_j) L_i^{(s)} \cos\theta_j \sin\theta_j \mathrm{d}\theta_j \mathrm{d}\phi_j. \tag{14}$$

The total reflected element of irradiance $\mathrm{d}W_r(\theta_j, \phi_j)$ is the sum of the components $\mathrm{d}W_r^{(p)}(\theta_j, \phi_j)$ and $\mathrm{d}W_r^{(s)}(\theta_j, \phi_j)$. Its expression is given by the sum of Eqs. (13) and (14), in which we replace $L_i^{(p)}$ and $L_i^{(s)}$ with $W_i/2\pi$ according to Eq. (11), and we replace $R_{jk}^{(p)}(\theta_j) + R_{jk}^{(s)}(\theta_j)$ with $2R_{jk}(\theta_j)$ according to the definition of the Fresnel reflection factor for natural light, Eq. (6). Therefore,

$$\mathrm{d}W_r(\theta_j, \phi_j) = R_{jk}(\theta_j) \frac{W_i}{\pi} \cos\theta_j \sin\theta_j \mathrm{d}\theta_j \mathrm{d}\phi_j. \tag{15}$$

The total reflected irradiance W_r is the sum of the element of irradiance reflected in all directions of the upper hemisphere:

$$W_r = \int_{\phi_j=0}^{2\pi} \int_{\theta_j=0}^{\pi/2} R_{jk}(\theta_j) \frac{W_i}{\pi} \cos\theta_j \sin\theta_j \mathrm{d}\theta_j \mathrm{d}\phi_j. \tag{16}$$

Since in Eq. (16) the integrated terms do not depend on ϕ_j, the integration according to ϕ_j yields a factor 2π. With $2\cos\theta_j \sin\theta_j = \sin 2\theta_j$, Eq. (16) becomes

$$W_r = W_i \int_{\theta_j=0}^{\pi/2} R_{jk}(\theta_j) \sin 2\theta_j \mathrm{d}\theta_j. \tag{17}$$

The ratio W_r/W_i then gives the diffuse reflectance r_{jk} of the interface[12]:

$$r_{jk} = \int_{\theta_j=0}^{\pi/2} R_{jk}(\theta_j) \sin 2\theta_j \mathrm{d}\theta_j. \tag{18}$$

Since the interface does not absorb light and since the energy is conserved, the transmittance t_{jk} of the interface for Lambertian incident light is

$$t_{jk} = 1 - r_{jk}. \tag{19}$$

B. Attenuation of Light in an Absorbing Medium

A collimated light flux traversing a path of length x in a nondiffusing absorbing medium, of linear absorption coefficient α, is attenuated according to a proportion t given by Beer's law

$$t = e^{-\alpha x}. \tag{20}$$

3. REFLECTANCE AND TRANSMITTANCE OF NONSCATTERING SUPERPOSED COLORING LAYERS OF DIFFERENT REFRACTIVE INDICES

In the present section, we characterize the reflection and transmission properties of a coloring (absorbing) nonscattering layer surrounded by two other media of different refractive indices. The difference of refractive index at the interfaces induces Fresnel reflections. Hence, multiple reflections occur within the considered layer, which increases the light absorption.

The nonscattering layer m_1 has a refractive index n_1, a wavelength-dependent absorption coefficient α_1, and a thickness h_1. Surrounding media m_0 and m_2 have respective refractive indices n_0 and n_2. The interface i_{01} between m_0 and m_1 and the interface i_{12} between m_1 and m_2 are parallel planes. The distance h_1 between these parallel planes is significantly larger than the wavelengths of light, thereby avoiding interference phenomena (Fig. 2).

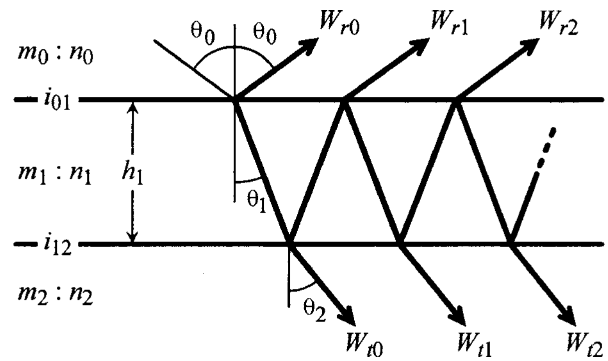

Fig. 2. Reflection and refraction within a nonscattering coloring layer m_1.

We distinguish the coloring layer alone, m_1, considered without its interfaces with the surrounding media, and the coloring layer with its interfaces, M_1, called the bounded layer. Inside the colored nonscattering layer m_1, light is either absorbed or transmitted, but not reflected. However, due to multiple reflections at the interfaces with surrounding media of different refractive indices, the bounded layer M_1 illuminated from medium m_0 reflects light back to medium m_0 and transmits light to medium m_2. The reflectance and the transmittance of the bounded layer M_1 are expressed according to the polarization of the incident light, according to its geometry (collimated or diffuse), and according to the side from which the bounded layer is illuminated (medium m_0 or medium m_2).

The incident light is incoherent and unpolarized (natural light), collimated with an incidence angle θ_0, and comes from medium m_0. The incident irradiance is decomposed into a p-polarized component and an s-polarized component. The p- and s-polarized components are reflected by the bounded layer, with proportions $R_{012}^{(p)}(\theta_0)$ and $R_{012}^{(s)}(\theta_0)$, respectively, and transmitted with proportions $T_{012}^{(p)}(\theta_0)$ and $T_{012}^{(s)}(\theta_0)$, respectively. The global reflectance and transmittance of the bounded layer illuminated by natural light is called $R_{012}(\theta_0)$, and its transmittance is called $T_{012}(\theta_0)$. For a Lambertian illumination from medium m_0, the bounded layer's reflectance and transmittance are called, respectively, r_{012} and t_{012}.

Let us express each of these reflectances and transmittances. They depend on the refractive indices n_0, n_1, and n_2, the coloring layer's thickness h_1 and its wavelength-dependent absorption coefficient α_1.

A. Reflectance and Transmittance of the Bounded Coloring Layer for Collimated Incident Light

A collimated incoherent and unpolarized light (natural light) illuminates the bounded layer M_1 from medium m_0 with an angle θ_0 (see Fig. 2). The incident irradiance W_i is decomposed into a p-polarized irradiance $W_i^{(p)}$ and an s-polarized irradiance $W_i^{(s)}$, with

$$W_i^{(p)} = W_i^{(s)} = W_i/2. \tag{21}$$

The components $W_i^{(p)}$ and $W_i^{(s)}$ are subjected to multiple reflections within the bounded layer M_1. To describe these multiple reflections, we calculate the reflected irradiances $W_r^{(p)}$ and $W_r^{(s)}$ and the transmitted irradiances $W_t^{(p)}$ and $W_t^{(s)}$. We then derive the reflectance and the transmittance of the bounded layer for the p-polarized component, the s-polarized component, and finally for natural incident light.

The phenomenon of multiple reflections is identical for the two components $W_i^{(p)}$ and $W_i^{(s)}$, with different Fresnel coefficients with respect to their respective polarizations. Hence, the interaction of polarized irradiances $W_i^{(p)}$ and $W_i^{(s)}$ with the bounded layer is presented only once, with a superscript * representing either superscript (p) or superscript (s).

A portion $R_{01}^*(\theta_0)$ of the polarized incident irradiance W_i^* is reflected by the interface i_{01}. It propagates along the specular direction, i.e., at an angle θ_0. The reflected irradance W_{r0}^* is the first contribution to the total polarizaed reflected irradiance W_r^*:

$$W_{r0}^* = R_{01}^*(\theta_0) W_i^*. \tag{22}$$

A portion $T_{01}^*(\theta_0)$ of W_i^* is transmitted into layer m_1 at an angle θ_1, crosses the layer along a path of length $h_1/\cos\theta_1$, and is attenuated by a factor $t_1(\theta_1)$ due to absorption (Beer's law):

$$t_1(\theta_1) = e^{-\alpha_1 h_1/\cos\theta_1}. \tag{23}$$

The irradiance reaching the interface i_{12} is therefore $T_{01}^*(\theta_0) t_1(\theta_1) W_i^*$. A portion $R_{12}^*(\theta_1)$ of this irradiance is internally reflected by the interface i_{12}, and a portion $T_{12}^*(\theta_1)$ is transmitted into medium 2 at an angle θ_2. According to Snell's law,

$$n_0 \sin\theta_0 = n_1 \sin\theta_1 = n_2 \sin\theta_2. \tag{24}$$

The irradiance W_{r0}^* is the first contribution to the total transmitted irradiance W_t^*:

$$W_{t0}^* = T_{01}^*(\theta_0) T_{12}^*(\theta_1) t_1(\theta_1) W_i^*. \tag{25}$$

The irradiance $T_{01}^*(\theta_0) R_{12}^*(\theta_1) t_1(\theta_1) W_i^*$ that is internally reflected by the interface i_{12} again crosses layer m_1 [attenuation factor $t_1(\theta_1)$ due to absorption] and is either internally reflected by the interface i_{01} [Fresnel reflection factor $R_{10}^*(\theta_1)$] or transmitted [Fresnel transmission factor $T_{10}^*(\theta_1)$] across that interface i_{01}. The irradiance W_{r1}^*, emerging from the interface i_{01}, is the second contribution to the total reflected irradiance W_r^*:

$$W_{r1}^* = T_{01}^*(\theta_0) R_{12}^*(\theta_1) T_{10}^*(\theta_1) t_1^2(\theta_1) W_i^*. \tag{26}$$

Then, owing to the multiple internal reflections, the light alternately crosses layer m_1 toward interfaces i_{01} and i_{12}. Along the same line of reasoning, we obtain obtained successive expressions of reflected in irradiances W_{rk}^* and transmitted irradiances W_{tk}^*. The sum of these irradiances forms respectively the total reflected irradiance W_r^* and the total transmitted iradiance irradiance W_t^*.

Let us first calculate the total reflected irradiance irradinace W_r^*. The irradiances W_{rk}^*, $k=1,2\ldots$, are internally reflected k times by the interface i_{12} and $k-1$ times by the interface i_{10}. They cross they the layer m_1 $2k$ times. Therefore, the generic expression of W_{rk}^*, for $k \geqslant 1$ is

$$W_{rk}^* = T_{01}^*(\theta_0)[R_{10}^*(\theta_1)]^{k-1}[R_{12}^*(\theta_1)]^k t_1^{2k}(\theta_1) T_{10}^*(\theta_1) W_i^*. \tag{27}$$

The total polarized reflected irradiance W_r^*, which emerges into medium m_0 with an angle θ_0, results from the sum of all the reflected irradiances W_{rk}^*:

$$W_r^* = R_{01}^*(\theta_0) W_i^* + \frac{T_{01}^*(\theta_0) T_{10}^*(\theta_1)}{R_{10}^*(\theta_1)} W_i^* \sum_{k=1}^{\infty} [R_{10}^*(\theta_1) R_{12}^*(\theta_1) t_1^2(\theta_1)]^k. \tag{28}$$

The infinite sum is a geometrical series. Since $T_{10}^*(\theta_1) = T_{01}^*(\theta_0)$ according to relation (9), Eq. (28) becomes

$$W_r^* = R_{01}^*(\theta_0)W_i^* + \frac{[T_{10}^*(\theta_1)]^2 R_{12}^*(\theta_1)t_1^2(\theta_1)}{1 - R_{10}^*(\theta_1)R_{12}^*(\theta_1)t_1^2(\theta_1)}W_i^*. \quad (29)$$

The ratio W_r^*/W_i^* gives the reflectance $R_{012}^*(\theta_0)$ of the bounded layer M_1 illuminated at an incidence θ_0 by a p-polarized or an s-polarized collimated incident light:

$$R_{012}^*(\theta_0) = R_{01}^*(\theta_0) + \frac{[T_{10}^*(\theta_1)]^2 R_{12}^*(\theta_1)t_1^2(\theta_1)}{1 - R_{10}^*(\theta_1)R_{12}^*(\theta_1)t_1^2(\theta_1)}. \quad (30)$$

For natural incident light, the total reflected irradiance W_r is given by the sum of the resulting p-polarized and s-polarized reflected irradiances given by Eq. (29):

$$W_r = W_r^{(p)} + W_r^{(s)} = R_{012}^{(p)}(\theta_0)W_i^{(p)} + R_{012}^{(s)}(\theta_0)W_i^{(s)}. \quad (31)$$

By replacing $W_i^{(p)}$ and $W_i^{(s)}$ with $W_i/2$ according to Eq. (21), we obtain

$$W_r = \tfrac{1}{2}[R_{012}^{(p)}(\theta_0) + R_{012}^{(s)}(\theta_0)]W_i. \quad (32)$$

The ratio W_r/W_i gives the reflectance $R_{012}(\theta_0)$ of the bounded layer M_1 surrounded by media m_0 and m_2, illuminated from m_0 by collimated natural light at an incidence θ_0:

$$R_{012}(\theta_0) = \tfrac{1}{2}[R_{012}^{(p)}(\theta_0) + R_{012}^{(s)}(\theta_0)]. \quad (33)$$

Let us now calculate the total transmitted irradiance W_t^* and derive the expression of the transmittance of the bounded layer for p-polarized, s-polarized, and natural incident light. The irradiances W_{tk}^* are internally reflected k times by the interface i_{12} and k times by the interface i_{01}. They cross the layer m_1 $2k+1$ times. The generic expression of W_{tk}^* for $k \geq 0$ is

$$W_{tk}^* = T_{01}^*(\theta_0)[R_{10}^*(\theta_1)]^k[R_{12}^*(\theta_1)]^k[t_1(\theta_1)]^{2k+1}T_{12}^*(\theta_1)W_i^*. \quad (34)$$

The total transmitted irradiance W_t^* that emerges into medium m_2 at an angle θ_2 results from the sum of all the transmitted irradiances W_{tk}^*,

$$W_t^* = T_{01}^*(\theta_0)T_{12}^*(\theta_1)t_1(\theta_1)W_i^*\sum_{k=0}^{\infty}[R_{10}^*(\theta_1)R_{12}^*(\theta_1)t_1^2(\theta_1)]^k, \quad (35)$$

where the infinite sum is a converging geometric series. Relation (35) becomes

$$W_t^* = \frac{T_{01}^*(\theta_0)T_{12}^*(\theta_1)t_1(\theta_1)}{1 - R_{10}^*(\theta_1)R_{12}^*(\theta_1)t_1^2(\theta_1)}W_i^*. \quad (36)$$

The ratio W_t^*/W_i^* gives the reflectance $T_{012}^*(\theta_0)$ of the bounded layer M_1 illuminated under an incidence θ_0 by a p-polarized or an s-polarized collimated incident light:

$$T_{012}^*(\theta_0) = \frac{T_{01}^*(\theta_0)T_{12}^*(\theta_1)t_1(\theta_1)}{1 - R_{10}^*(\theta_1)R_{12}^*(\theta_1)t_1^2(\theta_1)}. \quad (37)$$

For natural incident light, the total transmitted irradiance W_t is given by the sum of the p-polarized and the s-polarized transmitted irradiances $W_t^{(p)}$ and $W_t^{(s)}$, given by Eq. (36). We replace $W_i^{(p)}$ and $W_i^{(s)}$ by $W_i/2$ according to Eq. (21) and obtain

$$W_t = \tfrac{1}{2}[T_{012}^{(p)}(\theta_0) + T_{012}^{(s)}(\theta_0)]W_i. \quad (38)$$

The ratio W_t/W_i gives the transmittance $T_{012}(\theta_0)$ of the bounded layer M_1 surrounded with media m_0 and m_2, illuminated from m_0 by a collimated natural light with an incidence angle θ_0:

$$T_{012}(\theta_0) = \tfrac{1}{2}[T_{012}^{(p)}(\theta_0) + T_{012}^{(s)}(\theta_0)]. \quad (39)$$

Let us now assume that the collimated light comes from m_2 at an incidence angle θ_2. The description of the multiple reflections occurring within the bounded layer is the same as when the light comes from medium m_0. We obtain the same expressions of reflectance and transmittance as above with exchanged subscripts 0 and 2. For the p-polarized or the s-polarized incident irradiance components, the bounded layer's reflectance $R_{210}^*(\theta_2)$ is

$$R_{210}^*(\theta_2) = R_{21}^*(\theta_2) + \frac{[T_{12}^*(\theta_1)]^2 R_{10}^*(\theta_1)t_1^2(\theta_1)}{1 - R_{12}^*(\theta_1)R_{10}^*(\theta_1)t_1^2(\theta_1)}. \quad (40)$$

For natural incident light, the bounded layer's reflectance $R_{210}(\theta_2)$ is

$$R_{210}(\theta_2) = \tfrac{1}{2}[R_{210}^{(p)}(\theta_2) + R_{210}^{(s)}(\theta_2)], \quad (41)$$

where $R_{210}^{(p)}(\theta_2)$ and $R_{210}^{(s)}(\theta_2)$ are given by Eq. (40). Similarly, for the p-polarized or the s-polarized incident irradiance components, the bounded layer's transmittance $T_{210}^*(\theta_2)$ is

$$T_{210}^*(\theta_2) = \frac{T_{21}^*(\theta_2)T_{10}^*(\theta_1)t_1(\theta_1)}{1 - R_{12}^*(\theta_1)R_{10}^*(\theta_1)t_1^2(\theta_1)}, \quad (42)$$

and for natural incident light, its transmittance $T_{210}(\theta_2)$ is

$$T_{210}(\theta_2) = \tfrac{1}{2}[T_{210}^{(p)}(\theta_2) + T_{210}^{(s)}(\theta_2)]. \quad (43)$$

Since $T_{10}^*(\theta_1) = T_{01}^*(\theta_0)$ and $T_{12}^*(\theta_1) = T_{21}^*(\theta_2)$ according to relation (9), we observe that expressions (37) and (42) are identical. Thus, for the linearly polarized components, and thereby for natural light, the transmittance of the bounded layer M_1 does not depend on the orientation of light propagation:

$$T_{210}(\theta_2) = T_{012}(\theta_0). \quad (44)$$

The bounded layer's reflectance depends on the orientation of light propagation only in respect to the Fresnel reflection on the first interface encountered by the incident light, i.e., $R_{01}^*(\theta_0)$ in Eq. (30) and $R_{21}^*(\theta_2)$ in Eq. (40).

B. Diffuse Reflectance of the Bounded Coloring Layer

The diffuse reflectance r_{012} of the bounded coloring layer M_1 surrounded by nonscattering media m_0 and m_2 gives the fraction of incident Lambertian irradiance coming from medium m_0 that is reflected by the layer back to medium m_0.

The incident light is diffuse, incoherent, and unpolarized. It constitutes a Lambertian irradiance W_i, composed of elements of irradiance $\mathrm{d}W_i(\theta_0,\phi_0)$ that can be expressed as a function of the direction-independent radiance W_i/π (see Section 2):

$$\mathrm{d}W_i(\theta_0,\phi_0)=\frac{W_i}{\pi}\cos\theta_0\sin\theta_0\mathrm{d}\theta_0\mathrm{d}\phi_0. \tag{45}$$

Each element of irradiance $\mathrm{d}W_i(\theta_0,\phi_0)$ is reflected by the bounded layer M_1 with a proportion $R_{012}(\theta_0)$ given by Eq. (33). The term $R_{012}(\theta_0)$ accounts for the multiple reflections undergone by both linearly polarized components within the bounded layer M_1. The sum of all the reflected elements of irradiance $R_{012}(\theta_0)dW_i(\theta_0,\phi_0)$, performed over the all incidence angles (θ_0,ϕ_0) of the hemisphere, yields the total reflected irradiance W_r:

$$W_r=\int_{\phi_0=0}^{2\pi}\int_{\theta_0=0}^{\pi/2}R_{012}(\theta_0)\frac{W_i}{\pi}\cos\theta_0\sin\theta_0\mathrm{d}\theta_0\mathrm{d}\phi_0. \tag{46}$$

Since the integrated terms do not depend on the azimuth angle ϕ_0, the integration according to ϕ_0 yields a factor 2π. After rearranging the expression of W_r in the same manner as Eq. (17), the ratio W_r/W_i yields the diffuse reflectance r_{012} of the bounded layer M_1, surrounded by media m_0 and m_2 and illuminated from m_0 by diffuse light:

$$r_{012}=\int_{\theta_0=0}^{\pi/2}R_{012}(\theta_0)\sin 2\theta_0\mathrm{d}\theta_0. \tag{47}$$

This expression generalizes the diffuse reflectance r_{01} of an interface, Eq. (18), to a coloring nonscattering layer that has a refractive index different from its surrounding media.

C. Diffuse Transmittance of the Bounded Coloring Layer

The diffuse transmittance t_{012} of the bounded coloring layer M_1 surrounded by nonscattering media m_0 and m_2 gives the fraction of incident Lambertian irradiance coming from medium m_0 that is transmitted across the layer and therefore emerges into medium m_2.

Each incident element of irradiance $\mathrm{d}W_i(\theta_0,\phi_0)$ expressed in relation (45) is transmitted across the bounded layer M_1 with a proportion $T_{012}(\theta_0)$ given by Eq. (39). The sum of all transmitted elements of irradiance $T_{012}(\theta_0)\mathrm{d}E_i(\theta_0,\phi_0)$, performed over all the incidence angles (θ_0,ϕ_0) of the hemisphere, yields the transmitted irradiance W_t:

$$W_t=\int_{\phi_0=0}^{2\pi}\int_{\theta_0=0}^{\pi/2}T_{012}(\theta_0)\frac{W_i}{\pi}\cos\theta_0\sin\theta_0\mathrm{d}\theta_0\mathrm{d}\phi_0. \tag{48}$$

After applying the same simplifications as previously, we obtain the diffuse transmittance of the bounded coloring layer M_1:

$$t_{012}=\frac{W_t}{W_i}=\int_{\theta_0=0}^{\pi/2}T_{012}(\theta_0)\sin 2\theta_0\mathrm{d}\theta_0. \tag{49}$$

This expression generalizes the diffuse transmittance of t_{01} of an interface to an absorbing layer that has an index of refraction different from air. However, since the bounded coloring layer M_1 absorbs light, the energy is not conserved and t_{012} is different from $1-r_{012}$.

D. Reflectance and Transmittance of Two or More Superposed Nonscattering Bounded Coloring Layers

Two nonscattering coloring layers m_1 and m_2, of respective refractive indices n_1 and n_2, thicknesses h_1 and h_2, and absorption coefficients α_1 and α_2, are superposed and surrounded by nonscattering media m_0 and m_3. The three interfaces i_{01}, i_{12}, and i_{23} are parallel planes.

The reflectance and transmittance of the two superposed bounded layers, for collimated illumination from m_0 with an incidence angle θ_0, are called, respectively, $R_{0123}(\theta_0)$ and $T_{0123}(\theta_0)$. As previously, the incident light is incoherent and unpolarized.

The reflectance $R_{0123}(\theta_0)$ of two superposed bounded layers is the average of the two reflectances obtained for the p-polarized and the s-polarized components:

$$R_{0123}(\theta_0)=\tfrac{1}{2}[R_{0123}^{(p)}(\theta_0)+R_{0123}^{(s)}(\theta_0)]. \tag{50}$$

$R_{0123}^{(p)}(\theta_0)$ and $R_{0123}^{(s)}(\theta_0)$ are extensions of the reflectances $R_{012}^{(p)}(\theta_0)$ and $R_{012}^{(s)}(\theta_0)$ of a single coloring layer M_1 surrounded by media m_0 and m_2 [Eq. (30)]. When deriving the expression of $R_{012}^{*}(\theta_0)$, we considered the multiple reflections within the layer m_1 between the interfaces i_{01} [Fresnel reflection coefficient $R_{10}^{*}(\theta_1)$] and i_{12} [Fresnel reflection coefficient $R_{12}^{*}(\theta_1)$]. In the present case, we consider the multiple reflections within layer m_1 between the interfaces i_{01} [Fresnel reflection coefficient $R_{10}^{*}(\theta_1)$] and the bounded layer M_2 surrounded by media m_1 and m_3, of reflectance $R_{123}^{*}(\theta_1)$. The expression of $R_{0123}^{*}(\theta_0)$ therefore derives from the expression of $R_{012}^{*}(\theta_0)$, in which $R_{12}^{*}(\theta_1)$ is replaced by $R_{123}^{*}(\theta_1)$:

$$R_{0123}^{*}(\theta_0)=R_{01}^{*}(\theta_0)+\frac{[T_{01}^{*}(\theta_0)]^2R_{123}^{*}(\theta_1)t_1^2(\theta)}{1-R_{10}^{*}(\theta_1)R_{123}^{*}(\theta_1)t_1^2(\theta)}. \tag{51}$$

The thickness h_2 and the absorption coefficient α_2 of the layer m_2 are implicit within the term $R_{123}(\theta_1)$.

Similar considerations apply for the transmittance $T_{0123}(\theta_0)$:

$$T_{0123}(\theta_0)=\tfrac{1}{2}[T_{0123}^{(p)}(\theta_0)+T_{0123}^{(s)}(\theta_0)], \tag{52}$$

where the expressions $T_{0123}^{(p)}(\theta_0)$ and $T_{0123}^{(s)}(\theta_0)$ derive from the expressions $T_{012}^{(p)}(\theta_0)$ and $T_{012}^{(s)}(\theta_0)$ by replacing $T_{12}^{*}(\theta_1)$ with $T_{123}^{*}(\theta_1)$ in Eq. (37):

$$T_{0123}^{*}(\theta_0)=\frac{T_{01}^{*}(\theta_0)T_{123}^{*}(\theta_1)t_1(\theta_1)}{1-R_{10}^{*}(\theta_1)R_{123}^{*}(\theta_1)t_1^2(\theta_1)}. \tag{53}$$

By following the same line of reasoning but starting with the reflectance and transmittance of the bounded layer M_2 instead of the reflectance and transmittance of the bounded layer M_1, we obtain expressions that are exactly equivalent to expressions (51) and (53):

$$R_{0123}^{*}(\theta_0)=R_{012}^{*}(\theta_0)+\frac{[T_{012}^{*}(\theta_0)]^2R_{23}^{*}(\theta_2)t_2^2(\theta_2)}{1-R_{210}^{*}(\theta_1)R_{23}^{*}(\theta_2)t_2^2(\theta_2)} \tag{54}$$

and

$$T^*_{0123}(\theta_0) = \frac{T^*_{012}(\theta_0)T^*_{23}(\theta_2)t_2(\theta_2)}{1 - R^*_{210}(\theta_1)R^*_{23}(\theta_2)t_2^2(\theta_2)}, \quad (55)$$

with

$$t_2(\theta_2) = e^{-\alpha_2 h_2/\cos\theta_2}. \quad (56)$$

Along this line of reasoning, we express the reflectance and transmittance of k superposed bounded coloring layers $M_1, \ldots, M_k$ surrounded by media m_0 and m_{k+1}:

$$R_{0\ldots k+1}(\theta_0) = \tfrac{1}{2}[R^{(p)}_{0\ldots k+1}(\theta_0) + R^{(s)}_{0\ldots k+1}(\theta_0)], \quad (57)$$

$$T_{0\ldots k+1}(\theta_0) = \tfrac{1}{2}[T^{(p)}_{0\ldots k+1}(\theta_0) + T^{(s)}_{0\ldots k+1}(\theta_0)]. \quad (58)$$

The resulting expressions of $R^*_{0\ldots k+1}(\theta_0)$ and $T^*_{0\ldots k+1}(\theta_0)$ generalize Eqs. (54) and (55) to k stacked nonscattering coloring layers with distinct refractive indices:

$$R^*_{0\ldots k+1}(\theta_0) = R^*_{0\ldots k}(\theta_0) + \frac{[T^*_{0\ldots k}(\theta_0)]^2 R^*_{k,k+1}(\theta_k)t_k^2(\theta_k)}{1 - R^*_{k\ldots 0}(\theta_k)R^*_{k,k+1}(\theta_k)t_k^2(\theta_k)} \quad (59)$$

and

$$T^*_{0\ldots k+1}(\theta_0) = \frac{T^*_{0\ldots k}(\theta_0)T^*_{k,k+1}(\theta_k)t_k(\theta_k)}{1 - R^*_{k\ldots 0}(\theta_k)R^*_{k,k+1}(\theta_k)t_k^2(\theta_k)}, \quad (60)$$

with

$$t_k(\theta_k) = e^{-\alpha_k h_k/\cos\theta_k}. \quad (61)$$

If the incident light is Lambertian, we integrate expressions (57) and (58) over the hemisphere, as in the case of Eqs. (47) and (49), and obtain the diffuse reflectance $r_{0\ldots k+1}$ and transmittance $t_{0\ldots k+1}$ of the k superposed bounded coloring layers $M_1, \ldots, M_k$ surrounded by media m_0 and m_{k+1}:

$$r_{0\ldots k+1} = \int_{\theta_0=0}^{\pi/2} R_{0\ldots k+1}(\theta_0)\sin 2\theta_0 \mathrm{d}\theta_0 \quad (62)$$

and

$$t_{0\ldots k+1} = \int_{\theta_0=0}^{\pi/2} T_{0\ldots k+1}(\theta_0)\sin 2\theta_0 \mathrm{d}\theta_0. \quad (63)$$

4. REFLECTANCE OF NONSCATTERING COLORING LAYERS SUPERPOSED ON TOP OF A DIFFUSING MEDIUM

In the previous section, the reflectance and transmittance of stacked nonscattering layers have been expressed, both for collimated and for diffuse incident light, and for natural light or linearly polarized light. We now consider that the stacked layers are surrounded on one side by a transparent nonscattering medium and on the other side by a diffusing background. Each medium may have a distinct refractive index.

We express the reflectance of the background coated with colored nonscattering layers, i.e., the fraction of the incident irradiance that emerges from the interface i_{01}. The incident light is assumed to be collimated. The derived reflectance expressions are compatible with reflectance measurements performed with an integrating sphere by reference to the reflectance of a perfectly white diffuse reflector. For measurements performed with a radiance detector, we develop the expression of a reflectance factor, which takes into account the geometry of the capturing device.

First, we consider a single colored nonscattering layer m_1 of refractive index n_1, thickness h_1 and absorption coefficient α_1, surrounded on one side by a nonscattering medium m_0 of refractive index n_0, and on the other side by a diffusing background m_2 of refractive index n_2. The background is assumed to be a Lambertian reflector; i.e., it reflects a perfectly diffuse and unpolarized light. It is characterized by its wavelength-dependent intrinsic reflectance ρ_g.

After having established the reflectance of a single bounded nonscattering coloring layer M_1, we can replace it by any superposition of k bounded nonscattering coloring layers $M_1, \ldots, M_k$. In the special case where layer m_1 is surrounded on one side by a reflecting background of identical refractive index and on the other side by air, we obtain the classical Williams–Clapper model.[2]

A. Background Coated with One Nonscattering Layer

A collimated irradiance W_i (natural light) illuminates the interface i_{01} at an angle θ_0. It crosses the bounded coloring layer M_1 with an attenuation factor $T_{012}(\theta_0)$ [Eq. (39)] and penetrates into the diffusing background, where it is diffused, with a portion ρ_g being reflected. The reflected Lambertian irradiance w_1 is

$$w_1 = \rho_g T_{012}(\theta_0) W_i. \quad (64)$$

The reflected irradiance w_1 is unpolarized, owing to multiple scattering within the diffusing background, and can be assimilated to natural light. The bounded layer M_1 transmits a portion of w_1 into medium m_0 and reflects a portion r_{210} [Eq. (47)] toward the background m_2. The background reflects back toward the bounded layer m_1 a Lambertian irradiance w_2:

$$w_2 = \rho_g r_{210} w_1 = (\rho_g r_{210})\rho_g T_{012}(\theta_0) W_i. \quad (65)$$

The diffuse light is alternately reflected by the bounded layer M_1 and the background m_2. Since the background emits an irradiance $(\rho_g r_{210})^k \rho_g T_{012}(\theta_0) W_i$ at each internal reflection $k=0,1,2\ldots$, the total irradiance W_g emitted by the background toward the bounded layer M_1 is

$$W_g = \sum_{k=0}^{\infty} (\rho_g r_{210})^k \rho_g T_{012}(\theta_0) W_i. \quad (66)$$

The infinite sum yields a geometrical series that converges toward

$$W_g = \frac{\rho_g T_{012}(\theta_0)}{1 - \rho_g r_{210}} W_i. \quad (67)$$

The irradiance W_r that emerges into medium m_0 results from the transmission of the irradiance W_g across the bounded layer M_1. Since the bounded layer M_1 has a diffuse transmittance t_{210}, similar to that defined in Eq. (49), the emerging irradiance is

$$W_r = t_{210}\frac{\rho_g T_{012}(\theta_0)}{1-\rho_g r_{210}}W_i. \tag{68}$$

The reflectance R_{m1} of the background coated with the bounded layer m_1 is given by the ratio W_r/W_i:

$$R_{m1} = t_{210}\frac{\rho_g T_{012}(\theta_0)}{1-\rho_g r_{210}}. \tag{69}$$

B. Reflectance Factors

Often, the incident irradiance is not directly accessible. However, it is possible to measure it indirectly with a white reference support, whose reflectance spectrum is known and is generally close to 1 for all the wavelengths of the visible range. The irradiance W_r reflected by the sample and the irradiance W_{ref} reflected by the white reference support are captured by the same measuring device under the same illumination conditions. The ratio W_r/W_{ref} is called the reflectance factor.

The reflectance factor also depends on the geometry of the measuring device. An integrating sphere captures the reflected irradiance W_r completely. If the white reference support has a reflectance equal to 1, we have $W_{ref}=W_i$, and the expressions for the reflectance factor W_r/W_{ref} and the reflectance are identical.

The reflectance factor and the reflectance have different expressions if the capturing device is a radiance detector. The radiance detector does not capture the total reflected irradiance W_r but only the radiance $L_r(\theta_0')$ reflected in the direction θ_0' of the detector. This radiance $L_r(\theta_0')$ is defined by the flux emerging from a surface element ds of the sample within a solid angle $\mathrm{d}\Omega_0$:

$$L_r(\theta_0') = \frac{\mathrm{d}^2\Phi}{\mathrm{d}s\cos\theta_0\mathrm{d}\Omega_0}. \tag{70}$$

Let us consider the case of a diffusing background coated with a single nonscattering coloring layer m_1. We have shown that, owing to the multiple internal reflections, the background emits a total irradiance W_g, expressed in Eq. (67). We now derive, using the rules of radiometry,[11] the relation between the irradiance W_g and the radiance $L_r(\theta_0')$ captured by the radiance detector.

The flux $\mathrm{d}^2\Phi$ captured in the direction θ_0' corresponds to a collimated flux $\mathrm{d}^2\Phi_g$ emitted by the background m_2 in a direction θ_2' such that $n_0\sin\theta_0'=n_2\sin\theta_2'$. When crossing the bounded layer M_1, the flux is attenuated by a factor $T_{210}(\theta_2')$, equal to $T_{012}(\theta_0')$ according to relation (44). Therefore,

$$\mathrm{d}^2\Phi = T_{012}(\theta_0')\mathrm{d}^2\Phi_g. \tag{71}$$

Because of the refraction, the solid angle $\mathrm{d}\Omega_0$ containing the flux $\mathrm{d}^2\Phi$ and the solid angle $\mathrm{d}\Omega_2$ containing the flux $\mathrm{d}^2\Phi_g$ are different and related according to Eq. (10):

$$\mathrm{d}\Omega_2 = \left(\frac{n_0}{n_2}\right)^2\frac{\cos\theta_0'}{\cos\theta_2'}\mathrm{d}\Omega_0. \tag{72}$$

The flux $\mathrm{d}^2\Phi_g$ emitted by the background in the direction θ_2' within the solid angle $\mathrm{d}\Omega_2$, relative to the surface element ds, defines a radiance. Since the background is a Lambertian emitter of irradiance W_g, the radiance emitted into any direction is equal to W_g/π. Therefore,

$$\frac{W_g}{\pi} = \frac{\mathrm{d}^2\Phi_g}{\mathrm{d}s\cos\theta_2'\mathrm{d}\Omega_2}. \tag{73}$$

Let us express the reflected radiance $L_r(\theta_0')$ using Eqs. (67) and (70)–(73). First, in Eq. (71), we replace $\mathrm{d}^2\Phi_g$ by $(W_g/\pi)\mathrm{d}s\cos\theta_2'\mathrm{d}\Omega_2$ according to Eq. (73). The resulting expression of $\mathrm{d}^2\Phi$ is inserted into Eq. (70), which becomes

$$L_r(\theta_0') = \frac{W_g}{\pi}T_{012}(\theta_0')\frac{\cos\theta_2'\mathrm{d}\Omega_2}{\cos\theta_0'\mathrm{d}\Omega_0}. \tag{74}$$

In Eq. (74) we replace W_g with its expression (67) and replace the fraction on the right by $(n_0/n_2)^2$ according to Eq. (72). Equation (74) becomes

$$L_r(\theta_0') = \frac{1}{\pi}(n_0/n_2)^2T_{012}(\theta_0)T_{012}(\theta_0')\frac{\rho_g}{1-\rho_g r_{210}}W_i. \tag{75}$$

The ratio of the reflected radiance to the incident irradiance gives, by definition, the bidirectional reflectance distribution function (BRDF) of the background coated with a nonscattering coloring layer:

$$\mathrm{BRDF}(\theta_0,\theta_0') = \frac{L_r(\theta_0')}{W_i} = \frac{1}{\pi}(n_0/n_2)^2T_{012}(\theta_0')\frac{\rho_g}{1-\rho_g r_{210}}. \tag{76}$$

By measuring with the same illuminating and measuring conditions the radiance $L_{ref}=W_i/\pi$ reflected by a white reference support, we obtain the reflectance factor $R(\theta_0,\theta_0')$ of the background coated with the coloring layer, expressed as the ratio $L_r(\theta_0')/L_{ref}$:

$$R(\theta_0,\theta_0') = (n_0/n_2)^2T_{012}(\theta_0')\frac{\rho_g}{1-\rho_g r_{210}}. \tag{77}$$

The special case of computing the reflectance of a background coated with one scattering layer is identical to the problem of computing the exact reflectance of a varnished painting. One may verify that expression (77) is identical to the reflectance expression developed by Elias and Simonot [Ref. 9, p. 21, Eq. (2), but with different notation].

Note that in the expressions (69) and (77), the specular surface reflection component $R_{012}(\theta_0)$ has been discarded, as is the case in many photospectrometers.

C. Background Coated with Multiple Nonscattering Layers

The expression of the reflectance of a background coated with one nonscattering coloring layer m_1, Eq. (69), contains three terms relative to the bounded coloring layer: the transmittance $T_{012}(\theta_0)$ for a collimated illumination from medium m_0, the diffuse reflectance r_{210}, and the transmittance t_{210} for a diffuse illumination from medium m_2 [Eqs. (39), (47), and (49)]. We may extend Eq. (69) directly to the case of k superposed layers $m_{1\ldots k}$ by considering the superposed layers as a single bounded layer of transmittance $T_{0\ldots k+1}(\theta_0)$ for a collimated illumination from medium m_0 [Eqs. (58)], of diffuse reflectance $r_{k+1\ldots 0}$ and diffuse transmittance $t_{k+1\ldots 0}$ for a diffuse illumination from medium m_{k+1} [Eqs. (62) and (63) with exchanged subscripts 0 and $k+1$]. The reflectance $R_{m1\ldots k}$ of the background coated with k superposed layers $m_{1\ldots k}$ is therefore

$$R_{m1\ldots k} = t_{k+1\ldots 0}\frac{\rho_g T_{0\ldots k+1}(\theta_0)}{1-\rho_g r_{k+1\ldots 0}}. \qquad (78)$$

In analogy to expression (77), for a collimated illumination with incidence θ_0, for an observation angle θ_0', and for a perfectly white diffuse reflector as reference, the reflectance factor of the background coated with k superposed layers is

$$R(\theta_0,\theta_0') = (n_0/n_2)^2 T_{0\ldots k+1}(\theta_0) T_{0\ldots k+1}(\theta_0')\frac{\rho_g}{1-\rho_g r_{k+1\ldots 0}}. \qquad (79)$$

D. Particular Case: The Williams–Clapper Model

Because of the multiple internal reflections taking place at both sides of the interface in contact with the diffusing background, the reflectance model presented here is slightly different from the Williams–Clapper model.[2] In the Williams–Clapper model, the nonscattering layer m_1 and the diffusing background m_2 are assumed to have the same refractive index, i.e., $n_1=n_2$. Therefore, at the interface i_{12}, the Fresnel coefficients are $R_{21}(\theta)=R_{12}(\theta)=0$ (no reflection) and $T_{21}(\theta)=T_{12}(\theta)=1$ (total transmission). According to Eqs. (30), (33), (37), and (39)–(41), $R_{012}(\theta)$ reduces to $R_{01}(\theta)$, $R_{210}(\theta)$ becomes $R_{10}(\theta)t_1^2(\theta)$, and $T_{012}(\theta)$ becomes $T_{01}(\theta)t_1(\theta)$, with $t_1(\theta)$ given by Eq. (23). The term r_{210} becomes

$$r_{110} = \int_{\theta=0}^{\pi/2} R_{10}(\theta)t_1^2(\theta)\sin 2\theta \mathrm{d}\theta, \qquad (80)$$

and the term t_{210} becomes

$$t_{110} = \int_{\theta=0}^{\pi/2} T_{10}(\theta)t_1(\theta_1)\sin 2\theta \mathrm{d}\theta. \qquad (81)$$

By inserting these simplified expressions into Eq. (77), we retrieve the reflectance factor of the Williams–Clapper model, expressed for a collimated incidence and observation (respective angles θ_0 and θ_0'):

$$R(\theta_0,\theta_0') = (n_0/n_1)^2\frac{\rho_g T_{01}(\theta_0)T_{01}(\theta_0')t_1(\theta_1)t_1(\theta_1')}{1-\rho_g r_{110}}. \qquad (82)$$

In the case of a collimated illumination and an integrating sphere measuring geometry, the reflectance given by Eq. (69) reduces to

$$R_{m1} = t_{110}\frac{\rho_g T_{01}(\theta_0)t_1(\theta_1)}{1-\rho_g r_{110}}. \qquad (83)$$

We retrieve with this expression (83) the Shore–Spoonhower generalization[6] of the Williams–Clapper model for an integrating-sphere measuring geometry, also derived independently by Elias *et al.*[7]

E. Intrinsic Reflectance of the Background

The diffusing background is characterized by its intrinsic reflectance ρ_g. However ρ_g is not directly measurable, since the interface that separates the background (medium 2) and the air (medium 0) induces multiple internal reflections. However, it is possible to measure the reflectance of the background and derive its intrinsic reflectance ρ_g from Eq. (69) if the measurements are performed with an integrating sphere or from Eq. (77) if they are performed with a radiance detector.

The background, medium m_2, is directly in contact with the medium m_0. Thus, the absorption coefficient is $\alpha=0$; the terms $T_{012}(\theta_0)$, and $T_{012}(\theta_0')$ become, respectively, $T_{02}(\theta_0)$ and $T_{02}(\theta_0')$; and the terms r_{210} and t_{210} become, respectively, r_{20} and t_{20}. According to Eq. (69), the reflectance ρ of the background measured with an integrating sphere is

$$\rho = t_{20}\frac{\rho_g T_{02}(\theta_0)}{1-\rho_g r_{20}}, \qquad (84)$$

and according to Eq. (77), the reflectance factor of the background measured with a radiance detector is

$$R(\theta_0,\theta_0') = (n_0/n_2)^2 T_{02}(\theta_0)T_{02}(\theta_0')\frac{\rho_g}{1-\rho_g r_{20}}. \qquad (85)$$

The intrinsic reflectance ρ_g of the background can be obtained by inversion of Eq. (84)

$$\rho_g = \frac{\rho}{t_{20}T_{02}(\theta_0) + r_{20}\rho} \qquad (86)$$

or, respectively, by inversion of Eq. (85):

$$\rho_g = \frac{R(\theta_0,\theta_0')}{(n_0/n_2)^2 T_{02}(\theta_0)T_{02}(\theta_0') + r_{20}R(\theta_0,\theta_0')}. \qquad (87)$$

5. CONCLUSIONS

We propose a model for predicting the reflectance and transmittance of multiple stacked nonscattering coloring layers that have distinct refractive indices. The model relies on the modeling of the reflectance and transmittance of a bounded coloring layer, i.e., a coloring layer and its two interfaces with neighboring media of different refractive indices. By replacing within the expressions for the bounded layer reflectance (respectively, transmittance), the reflectance (respectively, transmittance) of a simple interface with the reflectance (respectively, transmittance) of a bounded layer, we are able to deduce the reflectance and transmittance of multiple stacked nonscattering layers of different refractive indices. This layer composition rule is then applied to deduce the reflectance of stacked nonscattering layers of distinct refractive indices superposed with a reflecting diffusing background that has its own refractive index. The Williams–Clapper model[2] as well as the air–paint[7] and the air–varnish–paint[9] reflection models become special cases of the proposed stacked layer model. Since the proposed model takes into account different illumination and measuring conditions, it is well suited for practical applications.

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