Figure 8.14: Explanation, in the continuous-world spectrum, of the sub-Nyquist artifact that occurs when sampling a continuous cosine function $g(x) = \cos(2\pi fx)$ whose frequency f is just slightly below half of the sampling frequency ($\frac{1}{2}f_s$). (a) The continuous-world spectrum $G(u)$ of our original cosine. (b) The continuous-world spectrum of the sampled cosine. Note that the spectrum (b) is an infinite replication of the original spectrum $G(u)$, where the replicas are centered about all the integer multiples of the sampling frequency f_s. Thanks to the first two impulse-pairs centered about its origin, the spectrum (b) of the sampled cosine basically corresponds to a sum of two cosines: our original continuous cosine, whose frequency is slightly below $\frac{1}{2}f_s$, and a new continuous cosine, whose frequency is slightly above $\frac{1}{2}f_s$. In order to formulate this claim more accurately, let us denote the sum of these two cosines (with halved amplitudes, so that the sum remains bounded between -1 and 1) by $g_1(x) = \frac{1}{2}\cos(2\pi[\frac{1}{2}f_s-\varepsilon]x) + \frac{1}{2}\cos(2\pi[\frac{1}{2}f_s+\varepsilon]x)$. The spectrum $G_1(u)$ of this sum of cosines is shown in the figure in a separate panel (a'). Now, the continuous-world spectrum (b) is also the spectrum of the sampled version of $g_1(x)$, using the same sampling frequency f_s. To see this, note that the spectrum (b) can be also considered as an infinite replication of the spectrum $G_1(u)$, where the replicas are located, once again, about all the integer multiples of f_s: Because the impulse pairs of every two neighbouring replicas of $G_1(u)$ fall exactly at the same points along the u axis, their halved amplitudes simply add up on top of each other, giving back precisely the spectrum (b). This means that the continuous-world spectrum (b) does not only belong to the sampled version of our original cosine function $g(x) = \cos(2\pi fx)$, but also to the sampled version of the cosine sum $g_1(x)$. This means, in turn, that the *sampled* version of our given cosine $g(x)$ is identical to the *sampled* version of the cosine sum $g_1(x)$ (although obviously $g(x)$ and $g_1(x)$ themselves are different). Now, based on Sec. D.3 of Appendix D, we know that the sum of two continuous cosines with slightly different frequencies gives a beating modulation effect. Thus, the sub-Nyquist artifact that appears when our original cosine $g(x)$ is being sampled (see Fig. 8.13(f)) is simply the sampled version of the beating modulation effect that occurs in the continuous cosine sum $g_1(x)$. (c), (d) Because the frequency of our cosine function is below $\frac{1}{2}f_s$, it can be perfectly reconstructed from its sampled version as stipulated by the sampling theorem, by multiplying the spectrum (b) with a rect function (a 1-valued pulse) extending from $-\frac{1}{2}f_s$ to $\frac{1}{2}f_s$, or, equivalently, by convolving the sampled version of the cosine signal with the corresponding sinc function. (e) When reconstructed by multiplying the spectrum (b) with a non-ideal substitute of the ideal rect function, debris of the new replicas that appeared in (b) due to the sampling may still subsist in the spectrum, causing a visible sub-Nyquist artifact. As shown in (b), this beating effect is generated by the sampling operation; but as we can see in (e), it becomes actually visible due to the non-ideal reconstruction. Note that the low beating frequency itself is not present in the spectrum, meaning that it is not a true moiré effect.

Legend for the improved version of **Figure 8.14** from the book: *Mastering the Discrete Fourier Transform in One, Two or Several Dimensions: Pitfalls and Artifacts*
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